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Yod bilan legirlangan qo'g'a tolasining fotoo'tkazuvchanlik va yarimo'tkazgichlik xususiyatlarini tadqiq etish

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Annotatsiya. Ushbu maqolada mahalliy iqlim sharoitida o'suvchi qo'g'a (*Typha*) o'simligi tolasining fizik va fotoelektrik xossalari eksperimental tadqiq etilgan. Tadqiqotda tolalarning fotoo'tkazuvchanlik xususiyatlarini o'zgartirish maqsadida yod bilan kimyoviy legirlash usuli qo'llanilgan. Yod bilan legirlangan namunalarning fotoo'tkazuvchanlik kinetikasi va spektral xususiyatlari o'rganilib, fototokning yoritish ta'sirida ortishi hamda yoritish to'xtatilgandan keyin relaksatsion jarayonlarning kechishi aniqlangan. Fotoo'tkazuvchanlik spektri tahlili asosida materialda chuqur energetik sathlar mavjudligi va ularning zaryad tashuvchilar hosil bo'lishidagi roli asoslangan. Spektrofotometrik tahlil natijalariga ko'ra, qo'g'a tolasining optik taqiqlangan zona kengligi $E_g = 2.36$ eV ekani aniqlangan. Olingan natijalar qo'g'a tolasining organik yarimo'tkazgich materiali sifatidagi potensialini baholash imkonini beradi.

Kalit so'zlar: qo'g'a (*Typha*) tolas, tabiiy tola, organik yarimo'tkazgich, fotoo'tkazuvchanlik, yod bilan legirlash, fototok, energetik sathlar, taqiqlangan zona, spektrofotometriya, optik xossalari.

Аннотация. В статье представлены результаты экспериментального исследования физических и фотоэлектрических свойств волокон рогоза (*Typha*), произрастающего в местных климатических условиях. С целью изменения фотоэлектрических характеристик волокна проведено химическое легирование йодом методом диффузионной пропитки. Исследованы кинетика фотопроводимости и спектральные характеристики легированных образцов. Установлено увеличение фототока под воздействием ультрафиолетового излучения, а также наличие релаксационных процессов после прекращения освещения. Анализ спектра фотопроводимости свидетельствует о наличии глубоких энергетических уровней, участвующих в процессах генерации и переноса носителей заряда. Методом спектрофотометрического анализа определена ширина оптической запрещенной зоны волокна рогоза, которая составляет приблизительно $E_g = 2,36$ эВ. Полученные результаты подтверждают наличие полупроводниковых и фотоактивных свойств волокна рогоза и позволяют рассматривать его в качестве перспективного природного органического полупроводникового материала.

Ключевые слова: волокно рогоза (*Typha*), природное волокно, органический полупроводник, фотопроводимость, легирование йодом, фототок, энергетические уровни, запрещенная зона, спектрофотометрия, фотоэлектрические свойства.

Abstract. This paper presents experimental studies of the physical and photoelectric properties of cattail (*Typha*) fibers under locally grown conditions. To modify the photoelectric properties of the fibers, chemical doping with iodine was performed using a diffusion impregnation method. The photoelectric conductivity kinetics and spectral characteristics of the iodine-doped fibers were investigated. An increase in photocurrent under ultraviolet irradiation and subsequent relaxation processes after termination of illumination were observed. Analysis of the photoelectric conductivity spectrum indicates the presence of deep energy levels associated with charge-carrier generation and transport in the material. The optical band gap of the cattail fiber was determined to be approximately $E_g = 2.36$ eV by spectrophotometric analysis. The obtained results demonstrate the semiconductor and photoactive characteristics of cattail fibers and indicate their potential as a natural organic semiconductor material for further research and possible applications in photoelectric and optoelectronic systems.

Keywords: cattail fiber (*Typha*), natural fiber, organic semiconductor, photoconductivity, iodine doping, photocurrent, energy levels, band gap, spectrophotometry, photoelectric properties.

I.Kirish

Bugungi kunda barqaror rivojlanish, resurslardan oqilona foydalanish va yashil iqtisodiyot tamoyillarining jadal rivojlanishi qayta tiklanuvchi xomashyolar asosida yangi funksional materiallarni yaratishga bo'lgan ilmiy va amaliy qiziqishni sezilarli darajada oshirmoqda. Ayniqsa, tabiiy tolalarning qayta tiklanuvchanligi, nisbatan past zichligi,

biologik parchalanishi, ekologik xavfsizligi hamda turli fizik-kimyoviy modifikatsiyalar yordamida ularning funksional xususiyatlarini boshqarish imkoniyati zamonaviy materialshunoslikning muhim yo'nalishlaridan biriga aylanmoqda. Tabiiy tolalar an'anaviy ravishda to'qimachilik, qurilish, kompozit materiallar va qadoqlash sanoatida qo'llanilgan bo'lsa-da, so'nggi yillarda

ularning funksional va elektroaktiv materiallar sifatidagi imkoniyatlarini o'rganishga ham katta e'tibor qaratilmoqda.

Shunday istiqbolli tabiiy xomashyolardan biri qo'g'a (*Typha*) o'simligidir. *Typha* turkumiga mansub o'simliklar namlik yuqori bo'lgan hududlarda keng tarqalgan bo'lib, yuqori biomassaga ega, tez o'suvchi va qayta tiklanuvchi tabiiy resurs sifatida qaraladi. So'nggi tadqiqotlarda qo'g'a bargi va poyasidan olinadigan tolalar tarkibida sellyuloza va gemitsellyuloza miqdori yuqori ekanligi, ularning strukturaviy tuzilishi esa ko'p bo'shliqli va g'ovak morfologiya bilan tavsiflanishi ko'rsatilgan [1, 2]. Bunday tuzilish qo'g'a tolalarining nisbatan past zichligi, yaxshi mexanik xususiyatlari va turli funksional materiallar yaratish uchun qulayligini ta'minlaydi. Adabiyotlarda qo'g'a tolalarining fizik-mexanik xususiyatlari ularning o'simlikning qaysi qismidan olingani, tolani ajratib olish usuli, kimyoviy ishlov berish sharoitlari, namlik miqdori va mikrostrukturaviy tuzilishiga sezilarli darajada bog'liqligi qayd etilgan. Xususan, suvli va ishqoriy ishlov berish usullari yordamida tolalar tarkibidagi mumsimon, ligninli va boshqa sellyuloza bo'lmagan komponentlarni kamaytirish hamda sellyuloza ulushini oshirish mumkin. Natijada tolaning yuzasi, kristallik darajasi, mexanik mustahkamligi va polimer matritsa bilan o'zaro ta'siri yaxshilanadi [2, 3]. Shu bilan birga, qo'g'a tolalarining xossalari turli *Typha* turlari, o'simlikning o'sish sharoiti va tolani qayta ishlash texnologiyasiga qarab sezilarli darajada farq qilishi mumkin. Qo'g'a tolalarining o'ziga xos morfologik tuzilishi ham ularning funksional xususiyatlarini belgilovchi muhim omildir. Tadqiqotlarda qo'g'a tolalarida ko'p bo'shliqli va g'ovak strukturalar mavjudligi, bunday tuzilma esa ularning yengilligi, suyuqliklarni yutish xususiyati, issiqlik va boshqa fizik jarayonlarda o'ziga xos xatti-harakatini ta'minlashi ko'rsatilgan [3-4]. Bundan tashqari, qo'g'a tolalari asosida kompozit materiallar, sorbentlar, izolyatsion materiallar, energiya saqlash tizimlari va boshqa funksional biomateriallarni yaratish imkoniyatlari ham o'rganilmoqda [1, 2]. Tabiiy tolalarning amaliy imkoniyatlarini kengaytirishning muhim yo'nalishlaridan biri

ularning fizik va elektr xususiyatlarini maqsadli modifikatsiya qilish hisoblanadi. Kimyoviy legirlash orqali material tarkibida qo'shimcha energetik sathlarning hosil bo'lishi, zaryad tashuvchilar konsentratsiyasining o'zgarishi va natijada fotoelektrik jarayonlarning boshqarilishi mumkin [5-8]. Qo'g'a tolalarining mexanik, morfologik va kompozit material sifatidagi xususiyatlari bo'yicha ilmiy adabiyotlarda yetarlicha ma'lumotlar mavjud bo'lsa-da, ularning elektron tuzilishi, fotoo'tkazuvchanligi, energetik sathlari va yorug'lik ta'siridagi zaryad tashuvchilar dinamikasi nisbatan kam o'rganilgan yo'nalishlardan biridir. Mavjud sharh tadqiqotlarida ham qo'g'a biomassa va tolalarining asosan mexanik, strukturaviy, ekologik hamda kompozit materiallardagi qo'llanilishiga e'tibor qaratilgan [1, 2]. Shu sababli qo'g'a tolalarining fotoelektrik xususiyatlarini chuqur o'rganish ushbu tabiiy materialning ilgari yetarlicha tadqiq etilmagan funksional imkoniyatlarini aniqlash nuqtai nazaridan dolzarb hisoblanadi.

Shu munosabat bilan, ushbu tadqiqotning asosiy maqsadi mahalliy iqlim sharoitida o'suvchi qo'g'a (*Typha*) o'simligi tolasining fizik va fotoelektrik xususiyatlarini eksperimental jihatdan o'rganish, shuningdek, yod bilan kimyoviy legirlashning materialning fotoo'tkazuvchanlik xususiyatlariga ta'sirini aniqlashdan iborat. Tadqiqot doirasida yod bilan legirlangan qo'g'a tolalarining fotoo'tkazuvchanlik kinetikasi, spektral xususiyatlari va energetik sathlari tahlil qilinib, materialning optik taqiqlangan zona kengligini aniqlashga alohida e'tibor qaratildi. Olingan natijalar qo'g'a tolalarining tabiiy, ekologik xavfsiz va funksional organik yarimo'tkazgich material sifatidagi potensialini baholash uchun ilmiy asos yaratadi.

II. MATERIAL VA USULLAR

Tadqiqot ob'ekti sifatida mahalliy iqlim sharoitida o'suvchi qo'g'a o'simligining tolalari tanlab olindi. Tolalarning yuzasidagi qo'shimchalarni yo'qotish maqsadida namunalar dastlab distillangan suvda bir necha bor yuvildi. So'ngra tolalar tarkibidagi namlikni bartaraf etish uchun namunalar termostatda 60⁰ C o'zgarmas haroratda

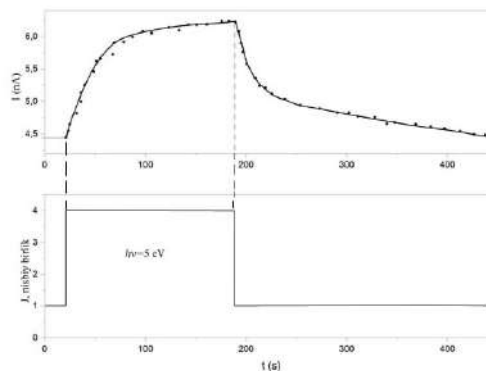
quritildi. Tolalarning fotoo'tkazuvchanlik xossalari maqsadli boshqarish va yangi energetik sathlar hosil qilish uchun kimyoviy legirlash usulidan foydalanildi [5]. Legirlovchi moddalar sifatida 5 % li yodning spirtidagi eritmasidan foydalanildi. Legirlash jarayoni diffuzion shimdirish usuli bilan amalga oshirildi. Bunda tolalar 6 soat davomida 80°C temperaturada legirlandi. Namunalarning fotoo'tkazuvchanlik spektrlarini yuqori aniqlikda o'lchash uchun yorug'lik manbai sifatida kvarts korpusli volfram spiralli lampa tanlandi. Ushbu lampa 0,16–4,35 mkm spektral diapazonda uzluksiz nurlanish hosil qilishi bilan tavsiflanadi va keng spektral sohada fotoo'tkazuvchanlikni tadqiq qilish imkonini beradi. Nurlanishni spektral tarkibiy qismlarga ajratish maqsadida Litiy ftorid (LiF) prizmasi bilan jihozlangan IKM-1 rusumli monoxromatordan foydalanildi. LiF prizmasi 0,12–8,5 mkm spektral diapazonda yuqori shaffoflikka ega bo'lib, ultrabinafsha, ko'rinadigan va infraqizil sohalarda monoxromatik nurlanishni shakllantirish imkonini beradi. Monoxromatordan chiqqan monoxromatik nur oqimini namunaga samarali yo'naltirish va uning intensivligini oshirish maqsadida chiqish qismiga LiF materialidan tayyorlangan yig'uvchi linza o'rnatildi. Tadqiq qilinayotgan namuna linzaning fokus tekisligiga joylashtirilib, uning butun ishchi yuzasi bo'ylab bir tekis yoritilishi ta'minlandi. Bu holat namuna bo'ylab fotoqo'zg'alish jarayonlarining bir xil kechishi hamda o'lchash natijalarining ishonchliligini oshirishga xizmat qildi. Tadqiqotlar asosan ,32–4,2 mkm spektral diapazonda olib borilganligi sababli monoxromatorning standart natriy xlorid (NaCl) prizmasi o'rniga Litiy ftorid (LiF) prizmasi o'rnatildi. Spektral shkalaning aniqligini ta'minlash maqsadida monoxromator ДПИИ-250 rusumli yuqori bosimli simob lampasi yordamida kalibrlandi. Kalibrlash natijasida spektral diapazondagi to'lqin uzunliklari yuqori aniqlik bilan belgilandi va eksperimental o'lchashlarning xatoligi minimallashtirildi.

Qo'g'a tolasining taqiqlangan zona kengligi miqdorini spektrofotometrik analiz orqali aniqlashda o'lchashlarni bajarish uchun SHIMADZU UV-2600 i plus UV-VIS spektrofotometridan foydalanildi.

Spektrofotometr 300 dan 1400 nm to'lqin uzunlikdagi diapazonda qattiq moddalarning diffuz qaytarish ko'effitsientini va namunalarning kolorimetrik xususiyatlarini o'rganishga mo'ljallangan qurilmadir. Ko'rsatilgan barcha to'lqin diapazonida o'lchovlarning nisbiy standart og'ishlari 0,1% dan 1% gachani tashkil qiladi.

III. NATIJALAR VA MUHOKAMA

Biz ushbu tadqiqotlarda yod bilan legirlangan qo'g'a tolalarining fotoo'tkazuvchanlik kinetikasini tadqiq qildik. Ultrabinafsha nur ($\lambda=254\text{ nm}$) bilan yoritilganda fototokning o'sishi va nur o'chirilganda fotoo'tkazuvchanlikning uzoq muddatli relaksatsiyasi kuzatildi. Bunda ichki fotoeffekt hodisasi namoyon bo'lganligi qayd qilindi.



1-rasm. Yod bilan legirlangan qo'g'a tolasining fotoo'tkazuvchanlik kinetikasi

1-rasmda yod bilan legirlangan qo'g'a tolasining 5 eV energiyali fotonlar ta'siridagi fotoo'tkazuvchanlik tokining vaqt bo'yicha o'zgarishi keltirilgan. Rasmning yuqori qismida fototokning vaqtga bog'liqligi, pastki qismida esa yoritish intensivligining nisbiy o'zgarishi tasvirlangan. Ko'rinib turibdiki, namunani yoritish boshlanishi bilan fototok tez ortib, dastlabki qiymati taxminan 4,45 nA dan 6,0 nA gacha ko'tariladi, keyin esa o'sish sur'ati sekinlashib, taxminan 6,1–6,2 nA qiymatda kvazistatsionar holatga yaqinlashadi. Yoritish davomida fototokning asta-sekin to'yinish holatiga o'tishi namunada fotogeneratsiya va zaryad tashuvchilarning yo'qolishi mexanizmlari o'rtasida muvozanat shakllanayotganidan dalolat beradi. Bunda zaryad tashuvchilarning rekombinatsiyasi, lokal energetik holatlarda tutilishi hamda

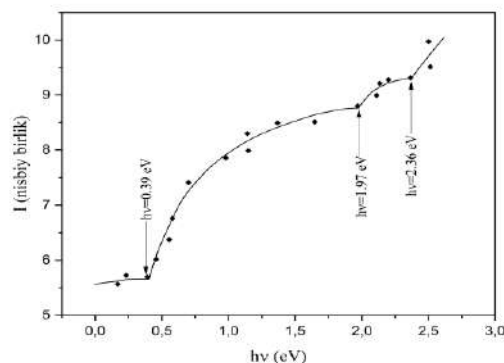
polimer zanjiri bo'ylab ularning ko'chishi bir vaqtning o'zida amalga oshadi. Taxminan 190 s da yoritish manbasining o'chirilishi bilan fototokning keskin pasayishi k Birinchi bosqichda tokning jadal k yoritish natijasida hosil bo'lgan tezki tashuvchilarning rekombinatsiyasi izohlanishi mumkin.

Umuman, 1-rasmda kuzatilgan yod bilan legirlangan qo'g'a fotoo'tkazuvchanlikning ikki xarakterga ega ekanini ko'rsatadi: vaqtida — fotogeneratsiya va tashuvchilarning to'planishi, to'xtatilgandan keyin esa — tezkor rekombinatsiya va sekin relaksatsion jarayonlar ustunlik qiladi. Shuningdek, 5 eV energiyali nurlanish ta'sirida fotooqimning qariyb 4,45 nA dan 6,2 nA gacha, ya'ni taxminan 1,4 marta ortishi mazkur materialning foto sezgirligini tasdiqlaydi. Olingan natijalar yod bilan legirlash polimer tolada zaryad tashuvchilar konsentratsiyasini oshirib, uning fotoelektrik xossalarini yaxshilashini ko'rsatadi.

2-rasmda yod bilan legirlangan qo'g'a tolasi fotoo'tkazuvchanlik spektri (yorug'lik spektri energiya diapazoni 0.3 eV dan 3.32 eV gacha) aniqlandi. Tadqiqotlar shuni ko'rsatdiki, namuna qorong'ida uzoq vaqt (1 soat) ushlab turilgandan keyin yorug'lik spektri yod bilan legirlangan qo'g'a tolasida $h\nu=0,39$ eV dan boshlab fotoo'tkazuvchanlik sezilarli darajada orta boshlagani kuzatildi. Foton energiyasi ortib borishi bilan yod uchun maksimal $h\nu=1.97$ eV da kuzatildi va yorug'lik nurining spektri taxminan $h\nu=2.36$ eV ga yetganda yana bir tok oshishi kuzatildi. Ushbu FO' spektrini qo'g'a tolasi taqiqlangan zona oralig'ining yuqori yarmida chuqur donor sathi mavjudligi bilan tushuntirish mumkin (2-rasm).

Bu shuni ko'rsatadiki, ushbu kimyoviy elementlar bilan legirlangan qo'g'a tolasi namunasining fotoo'tkazuvchanligi asosan zonalararo va zonalar orasidagi energetik sathlarda elektron-teshik juftligini hosil qilish jarayoni bilan bog'liq. Yarimo'tkazgichli organik materiallarda fotoo'tkazuvchanlik hodisasi yorug'lik fotonlarining material strukturasidagi elektronlar bilan o'zaro

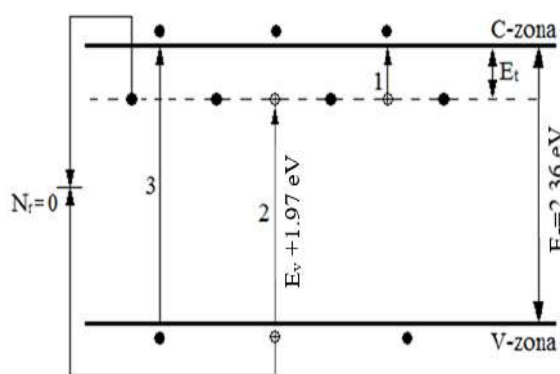
ta'sirlashuvi natijasida elektron-teshik juftliklari (eksitonlar) hosil bo'lishi bilan izohlanadi.



2-rasm. Yod bilan legirlangan qo'g'a tolasiining fotoo'tkazuvchanlik spektri

Bu faqat ushbu elementlar qo'g'a tolasi namunasining taqiqlangan zona oralig'ida chuqur sath hosil qilgan taqdiridagina amalga oshadi. Yod elementi taqiqlangan zona oralig'ining yuqori yarmida chuqur donor sath hosil qiladi deb faraz qilsak, FO' spektrining boshlanishi ($h\nu=0,39$ eV) qo'g'a tolasi tarkibidagi yodning ionlanish energiyasi, taqiqlangan zona kengligida $E_c-0,39$ eV chuqur sathni hosil qilganligi bilan bog'liq. Agar yod bilan legirlangan qo'g'a tolasi n -tipi o'tkazuvchanlikka ega ekanligini hisobga olsak, u holda, $E_t=E_c-0,39$ eV bo'lgan sath, taqiqlangan zona kengligining yuqori yarmida joylashgan deb aytish mumkin. FO' spektri tahlili $E_t=E_c-0,39$ eV bo'lgan sath donorlik xususiyatiga ega ekanligini ko'rsatadi. $0,39 \leq h\nu \leq 1.97$ eV sohasida bir xil ravishda uzluksiz ortib boradi, yorug'lik kuchaysa, moddada ko'proq foton molekullar bilan ta'sirlashadi. Natijada ko'proq elektronlar yoki zaryad tashuvchilar hosil bo'ladi $\rightarrow$ FO' o'sadi. Agar toladagi struktura ochiq, tartibli va bog'lamli bo'lsa, hosil bo'lgan zaryadlar oson harakatlanadi. Bu ham uzluksiz o'suvchi tokni ta'minlaydi. Energiya kam bo'lsa, yoki energiyasi past bo'lsa, fotogeneratsiya qilingan elektron-kovak juftlari darhol harakatga keladi. Yorug'lik ta'sirida elektron va kovaklarning uzluksiz hosil bo'lib, harakatlanishi; energiya kamligi yoki oson ajralishi; strukturaning tartiblili va zaryad tashuvchilar mobilligi natijasida ro'y beradi. FO' bir me'yorda uzluksiz o'sishi elektronlarning E_t sathidan C-zonasiga o'tishi

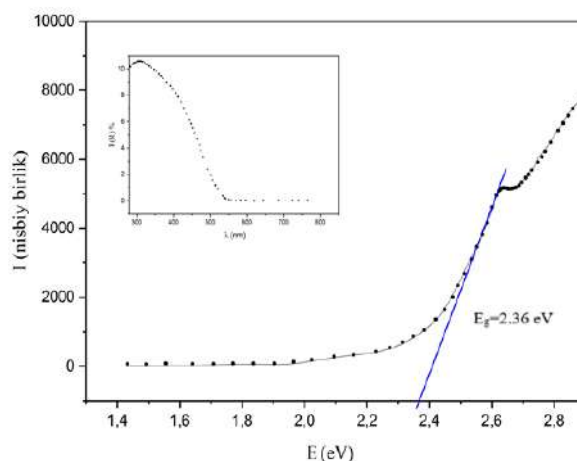
bilan bog'liq. Fotoo'tkazuvchanlikning $h\nu \geq 1.97$ eV dan boshlab yana o'sishi elektronlarning V-zonadan E_t sathga o'tishi bilan bog'liq. Fototokning $h\nu = 2.36$ eV dan boshlanadigan o'sishi elektronlarning V-zonadan to'g'ridan to'g'ri C-zonaga o'tishi bilan bog'liq. Olingan natijalarni chuqur sath mavjud bo'lgan yarimo'tkazgich materiallarning umumiy qabul qilingan ma'lumotlar bilan solishtirganda, qo'g'a tolasi optik taqiqlangan zona kengligining qiymati $E_g = 0.39 + 1.97 = 2.36$ eV ga teng ekanligi kelib chiqadi.



3-rasm. Yod bilan legirlangan qo'g'a tolasining zona diagrammasi

Ushbu qiymat qo'g'a tolasi spektrofotometrik analiz taxlili bilan ham tasdiqlangan.

Qo'g'a tolasining spektrofotometrik tahlili deganda tolaning elektromagnit nurlanish (UB, ko'rinadigan va IQ) bilan o'zaro ta'sirini o'rganish orqali uning optik, kimyoviy va energetik tuzilishini miqdoriy baholash tushuniladi. Qo'g'a tolasining taqiqlangan zona qiymatini optik usul bilan aniqlashni diffuz qaytish elektron spektroskopiyasi metodi orqali amalga oshirildi. Ma'lumki, elektro'tkazuvchi polimer materiallarning elektrokimyoviy xususiyatlari ularning elektron tuzilishiga bog'liq bo'lib, bu taqiqlangan zona qiymati bilan tavsiflanadi. Namunaning elektron tuzilishini tahlil qilish uchun diffuz qaytarish elektron spektroskopiyasi yordamida amalga oshirildi. Diffuz qaytarish elektron spektroskopiyasi tahlili natijalari Taus tenglamasi asosida topilgan. Taus grafigi to'g'ridan-to'g'ri zonalararo o'tishlarini baholash uchun qo'llanildi



4-rasm. Yod bilan legirlangan qo'g'a tolasi uchun $F(R)$ ning to'liq uzunligi (ichki rasm) va Taus egri chiziqlariga spektral bog'liqligi

Sof qo'g'a tolasi uchun $h\nu - (F(R)h\nu)^2$ bog'liqligini ekstrapolyatsiya qilish orqali topilgan taqiqlangan zona qiymati $E_g \approx 2.36$ eV ni tashkil qilishi aniqlandi. Bu esa o'rganilgan namunaning yarimo'tkazgich xossasiga egaligidan, hamda uning ko'rinadigan nur sohasida fotokatalitik faollikni namoyon qilish bo'yicha yuqori potentsialga egaligini ko'rsatadi.

IV.XULOSA.

O'tkazilgan eksperimental tadqiqotlar natijasida qo'g'a (*Typha*) tolasining fotoelektrik va yarimo'tkazgichlik xususiyatlari aniqlandi. Yod bilan legirlash natijasida tolaning fotoo'tkazuvchanligi sezilarli darajada oshishi, fototokning yoritish jarayonida o'sishi va yoritish manbai o'chirilgandan keyin rekombinatsion hamda relaksatsion jarayonlarning kechishi kuzatildi. Fotoo'tkazuvchanlik spektri tahlili qo'g'a tolasida chuqur energetik sathlar mavjudligini ko'rsatdi.

Spektral tahlil natijalaridan qo'g'a tolasining optik taqiqlangan zona kengligi $E_g = 2.36$ eV ekanligi aniqlanib, ushbu qiymat spektrofotometrik usulda olingan natijalar bilan tasdiqlandi. Shu asosda yod bilan legirlangan qo'g'a tolasi fotoaktiv va yarimo'tkazgich xususiyatlarga ega bo'lgan istiqbolli tabiiy material sifatida baholanishi

mumkin. Olingan natijalar uning organik elektronika, fotoelektrik qurilmalar va fotokatalitik jarayonlarda qo'llanish

imkoniyatlarini o'rganish uchun ilmiy asos yaratadi.

Adabiyotlar

- [1]. Tarus, B. K., Alfredy, T. S., Mwasiagi, J. I., Jande, Y. A. Ch. Valorisation of cattail (Typha) biomass: Fibre extraction, properties, and applications in sustainable material systems. *Bioresource Technology Reports*, 2026, Vol. 33, 102570.
- [2]. Parvin, S. et al. Harnessing Cattail Biomass for Sustainable Fibers and Engineered Bioproducts: A Review. *Global Challenges*, 2025.
- [3]. Wu, S., Zhang, J., Li, C. et al. Characterization of potential cellulose fiber from cattail fiber: A study on micro/nano structure and other properties. *International Journal of Biological Macromolecules*, 2021, Vol. 193, pp. 27–37. DOI: 10.1016/j.ijbiomac.2021.10.088.
- [4]. Moghaddam, M. K. Typha Leaves Fiber and Its Composites: A Review. *Journal of Natural Fibers*, 2022, Vol. 19, No. 13, pp. 4993–5007. DOI: 10.1080/15440478.2020.1870643.
- [5]. A.T.Мамадалимов, З.А.Додобоев, К.Махмудов, Ш.Туйчиев, М.Шерматов.

- Фотопроводимость хлопковых волокон. Доклады АН РУз 1995г. №5-6. стр.22-24.
- [6]. Mamadalimov A.T., Muminova G.M., Gulamov Sh.A., Naimov E.G., Xudayberdiyeva X.U., Alimova E.A., Komilova D.T. Electrophysical properties of surface – treated natural plant fibers // *Advanced Materials for Optics and Photonics: Chemistry and Engineering Perspectives (AMOP 2025)*. SPIE 2025. Vol.14014, <https://doi.org/10.1117/12.3095529> (Scopus)
- [7]. Mamadalimov A.T., Khakimova N.K., Muminova G.M., Isakov Q.K., Inkhonova A., Azimova D.A. Investigation of the optoelectronic properties of natural plant – derived typha surfaces treated with chemical elements // *Advanced Materials for Optics and Photonics: Chemistry and Engineering Perspectives (AMOP 2026)*. SPIE 2026. Pp. 1-6 (Scopus)
- [8]. Xakimova N.K., Mo'minova G.M. Yuzasiga ishlov berilgan tabiiy o'simliklardan qo'g'a tolalarining elektrofizik xossalari // *Ilmiy habarnoma. Andijon davlat universiteti. fizika va matematika tadqiqotlari* 2026. 13-18-b.

Комплексное исследование монокристалла кремния, легированного редкоземельными элементами Tb или Tm.

УДК 612.315.592.3

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Annotatsiya: Ushbu ishda nodir yer elementlari — terbiy (Tb) va tuli (Tm) bilan legirlangan kremniy monokristallning elektrofizik, strukturaviy hamda spektral xususiyatlariga ta'siri bo'yicha eksperimental tadqiqot natijalari keltirilgan. KEF-40, KEF-7 va KDB-20 markali kremniy namunalarda 1250 °C haroratda yuqori haroratli diffuzion tavlaniş jarayoni amalga oshirildi. Tadqiqot natijalari shuni ko'rsatdiki, Tb va Tm bilan legirlash kremniyning solishtirma elektr qarshiligi va zaryad tashuvchilar kontsentratsiyasida sezilarli o'zgarishlarni yuzaga keltiradi. Xususan, Tm elementi bilan legirlash natijasida p-turdagi o'tkazuvchanlikning n-turga o'tishi — ya'ni kompensatsiya hodisasi kuzatildi. Nodir yer elementlarining kremniy matritsasiga yuqori darajada kiritilganligi neytron-aktivatsion tahlil orqali eksperimental tarzda tasdiqlandi. Fourier-IK spektroskopiyasi natijalari legirlash jarayonida optik faol kislorod kontsentratsiyasi samarali kamayganini ko'rsatdi. Raman spektroskopiyasi esa kremniy panjarasida strukturaviy deformatsiya paydo bo'lishini va yangi fonon rejimlarining shakllanishini aniqladi. Bu holat legiruvchi atomlar bilan kremniy panjarasi orasidagi faol o'zaro ta'sir mavjudligini tasdiqlaydi. Olingan natijalar kremniy materialining fizik xususiyatlarini yaxshilash va uni zamonaviy yarimo'tkazgich texnologiyalarida qo'llash uchun yangi istiqbollarni ochib beradi.

Kalit so'zlar: kremniy monokristali, legirlash, nodir yer elementlari, terbiy (Tb), tuli (Tm), yuqori haroratli diffuziya, elektr qarshiligi, neytron-faollashtirish tahlili, Raman spektroskopiyasi, optik faol kislorod, nuqson tuzilmalari, yarim o'tkazgich materiallar.

Аннотация: В работе представлены результаты экспериментального изучения влияния легирования монокристаллического кремния редкоземельными элементами — тербием (Tb) и тулием (Tm) — на его электрофизические, структурные и спектральные характеристики. Образцы кремния марок КЭФ-40, КЭФ-7 и КДБ-20 подвергались высокотемпературному диффузионному отжигу при 1250 °C с последующим быстрым охлаждением. Выявлено, что легирование Tb и Tm приводит к существенному изменению удельного сопротивления и концентрации носителей заряда, причём в случае тулия отмечается перекомпенсация типа проводимости с дырочного на электронный. Нейтронно-активационным методом экспериментально подтверждена высокая степень внедрения атомов редкоземельных элементов в кремниевую матрицу. Методами Фурье ИК-спектроскопии установлено, что легирование эффективно снижает концентрацию оптически активного кислорода, а результаты Раман-спектроскопии указали на возникновение структурных напряжений и формирование новых фоновых мод, подтверждающих активное взаимодействие легирующих элементов с кремниевой решёткой. Полученные данные открывают перспективы для улучшения свойств кремния и его применения в полупроводниковых технологиях.

Ключевые слова: монокристаллический кремний, легирование, редкоземельные элементы, тербий (Tb), тулий (Tm), высокотемпературный отжиг, удельное сопротивление, нейтронно-активационный анализ, Раман-спектроскопия, оптически активный кислород, дефектные структуры, полупроводниковые материалы.

1. ВВЕДЕНИЕ.

Современные тенденции развития микроэлектроники и полупроводниковых технологий требуют создания материалов с улучшенными и стабильно воспроизводимыми электрофизическими характеристиками. В частности, особое внимание уделяется кремнию, являющемуся основным материалом в производстве полупроводниковых устройств, от качества которого зависят эксплуатационные характеристики конечных изделий [1-5]. Одной из эффективных стратегий модификации

свойств кремния является легирование редкоземельными элементами (РЗЭ), которые обладают уникальными физико-химическими характеристиками и способны заметно изменять его электронные и структурные параметры [3].

Актуальность изучения влияния редкоземельных элементов, таких как тербий (Tb) и тулий (Tm), на кремний обусловлена их способностью не только контролировать концентрацию и подвижность носителей заряда, но и значительно уменьшать содержание нежелательных примесей, например

оптически активного кислорода, что критически важно для повышения эффективности и стабильности работы микроэлектронных компонентов [4]. Кроме того, установлено, что внедрение РЗЭ вызывает образование локальных напряжений и дефектных структур в кремниевой решётке, что активно влияет на фоновые и электронные свойства материала, открывая возможности для разработки новых полупроводниковых приборов с заданными параметрами [6-8].

Несмотря на значительный интерес к подобным исследованиям, механизмы влияния редкоземельных элементов на свойства кремния остаются недостаточно изученными, особенно в условиях различных технологических режимов обработки, таких как высокотемпературный отжиг с последующим быстрым охлаждением. Именно это обуславливает актуальность проведённого исследования, целью которого является изучение электрофизических и структурных изменений в монокристаллическом кремнии при легировании тербием и тулием с использованием современных методов спектроскопии и нейтронно-активационного анализа.

Полученные в настоящей работе результаты вносят существенный вклад в понимание физико-химических процессов, происходящих при взаимодействии редкоземельных элементов с кремниевой матрицей, и открывают перспективы дальнейшего совершенствования технологий получения полупроводниковых материалов с улучшенными функциональными характеристиками.

II. МАТЕРИАЛЫ И МЕТОДЫ

Для исследования влияния редкоземельных элементов на основные

электрофизические параметры монокристаллического кремния, были отобраны образцы кремния марки КЭФ-40, КЭФ-7 и КДБ-20. Которые были подвергнуты механической и химической обработки до получения металлического слоя легируемых металлов, тулия и тербия, с помощью вакуумного напыления на установке ВУП-4, источников металлов были гранулы с чистотой 99,999. После получения металлического слоя легируемых металлов образцы были помещены в кварцевые трубки для получения вакуума до 10^{-4} мм.рт.ст., после чего концы трубок запаивались. После подготовки ампулы с образцами помещались в муфельный печь для прохождения высокотемпературного отжига. Отжиг проводился при 1250 °С в течение 10 и 50 часов. Для сохранения термодинамической неравновесной состоянии, было проведено быстрая закалка образцов, в масле, температура которого была равна 4–5 °С.

III. РЕЗУЛЬТАТЫ И ОБСУЖДЕНИЕ

После диффузионного отжига, с помощью метода Ван дер Пау, на установке ЕСОPIA HMS-3000, были измерены основные электрофизические параметры образцов до и после диффузионного отжига, результаты которых представлены в таблицах 1–3. В таблице 1 и 2 представлены результаты для образцов кремния марки КЭФ-40 и КДБ-20, ТО (термообработанные) и легированные Tb или Tm, при 1250 °С в течение 50 часов с последующей быстрой закалкой. В таблице 3 приведены результаты для образцов кремния марки КЭФ-7,5, ТО (термообработанные) и легированные Tb, при 1250 °С в течение 10 часов с последующей быстрой закалкой.

Таблица 1.

Параметр	До диффузии	После диффузии		
		ТО	Tb	Tm
ρ (Ом·см)	41,4	47,0	30,3	37,5
μ (см ² /В·с)	1406,9	983,9	341,5	293,0
n (см ⁻³)	$1,07 \cdot 10^{14}$	$1,35 \cdot 10^{14}$	$6,04 \cdot 10^{14}$	$5,69 \cdot 10^{14}$

Таблица 2.

Параметр	До диффузии	После диффузии		
		ТО	Тb	Tm
ρ (Ом·см)	20,3	140,1	33,0	14,2
μ (см ² /В·с)	418,7	173,0	217,7	1314,9
n (см ⁻³)	$7,35 \cdot 10^{14}$	$2,58 \cdot 10^{14}$	$8,71 \cdot 10^{14}$	$3,36 \cdot 10^{14}$

Таблица 3.

Параметр	До диффузии	После диффузии	
		ТО	Тb
ρ (Ом·см)	15,67	23,4	16,8
μ (см ² /В·с)	686	560	683
n (см ⁻³)	$6,1 \cdot 10^{14}$	$5,7 \cdot 10^{14}$	$5,6 \cdot 10^{14}$

Как показывают результаты представленные в таблицах 1-3, удельное сопротивление легированных образцов, в случае, кремния марки КЭФ-40, уменьшается от 41,4 до 30,3 при легирования Тb и до 37,5 при легировании Тm. В образцах кремния марки КДБ-20 видно повышение удельного сопротивления до 33 для Тb, а в образцах легированных Тm, наблюдается уменьшение удельного сопротивления до 14,2 и изменение типа проводимости с дырочного в электронный. В образцах же кремния марки КЭФ-7,5 наблюдается малое увеличение, до 16,8 Ом·см. Это может быть связано разницей по

длительности высокотемпературного отжига, так как в кристаллической решетке кремния при 50 часовом отжиге, концентрация образцованных термодинамически неравновесных состояний, дефектных структур или комплексов больше чем в образцах, которые отжигались в течение 10 часов. И как видно тулий при легировании в кремний, образуя соединения, которые выступают в роли доноров, увеличивает концентрацию неосновных носителей, для дырочного кремния, и производит перекомпенсацию основных носителей заряда, то есть дырок.

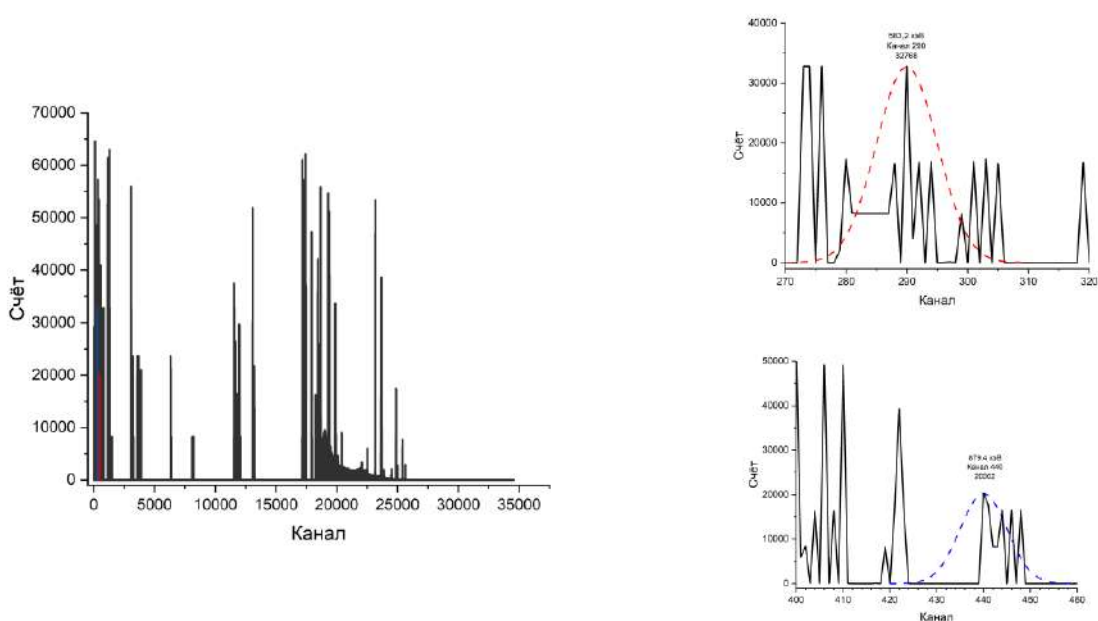


Рис.1. Гамма-спектр образца n-Si<Tb> 1250 °С, 10 ч., быстрое охлаждение

Для определения количества помощью нейтронно-активационного внедренных атомов тербия и тулия с анализа, образцы были подвергнуты

облучению тепловыми нейтронами с энергиями $E_n = 0.025 \div 5$ эВ, интенсивность потока была равна $I_n = 5 \cdot 10^{13}$ нейтр.·см⁻²·с⁻¹ и доза облучения составляло $D_n = 60 \div 80$ мкЗв/час. После облучения с помощью сверхчистого германиевого детектора GC2018 были получены гамма спектры от образцов. Полученный гамма-спектр для кремния представлены на рис. 1.

С помощью полученных спектров и базы данных гамма-спектров [3, 6] изотопов Tb¹⁶⁰ (583,2 кэВ и 879,4 кэВ) и Tm¹⁷⁰ (84,3

кэВ и 367,9 кэВ) был произведен количественный анализ, результат которого представлен в таблице 4.

По результатам анализа было определено что, в образцах кремния, прошедших термообработку в течение 10 часов и последующей быстрой закалкой содержится 0,026 мг тербия, а в образцах с медленным охлаждением 0,027 мг, а в образцах с 50 часовым отжигом и быстрой закалкой содержится 0,015 мг тербия и для образцов легированных тулием 0,011 мг тулия. Что указывает на успешное легирование кремния тулием и тербием.

Таблица 4.

№	Образец	Масса, мг	Содержание %, от. масс		Концентрация, атом/см ³	
			Tb	Tm	Tb	Tm
1.	n-Si<Tb> 1250 °C, 10 ч., быстрое охлаждение	68,3	$3,8 \times 10^{-2}$	-	$9,8 \times 10^{16}$	-
2.	n-Si<Tb> 1250 °C, 10 ч., медленное охлаждение	67,2	$4,1 \times 10^{-2}$	-	$1,04 \times 10^{17}$	-
3.	n-Si<Tb> 1250 °C, 50 ч., быстрое охлаждение	145,3	$1,7 \times 10^{-2}$	-	$9,3 \times 10^{16}$	-
4.	p-Si<Tb> 1250 °C, 50 ч., быстрое охлаждение	149,0	$1,0 \times 10^{-2}$	-	$5,61 \times 10^{16}$	-
5.	n-Si<Tm> 1250 °C, 50 ч., быстрое охлаждение	149,9	-	$1,0 \times 10^{-3}$	-	$5,64 \times 10^{16}$
6.	p-Si<Tm> 1250 °C, 50 ч., быстрое охлаждение	152,0	-	$7,0 \times 10^{-3}$	-	$3,8 \times 10^{17}$

Как известно [9-10], РЗЭ используются для, так называемой, очистки кремния от фоновых примесей, в частности оптически активного кислорода, нами были проведены работы по оценке концентрации оптически активного кислорода и углерода с помощью Фурье ИК-спектрометра ФСМ-2201. Полученные спектры представлены на рис. 1–3.

Как видно с рис. 1 и 2, при легировании кремния тулием и тербием, в течении 50 часов концентрация оптически активного кислорода уменьшается, для n-типа образцов с $6,64 \times 10^{17}$ см⁻³ до $2,08 \times 10^{17}$ см⁻³ в образцах легированных тербием и до $6,3 \times 10^{17}$ см⁻³ в образцах легированных тулием. В образцах p-типа наблюдается уменьшение с $7,7 \times 10^{17}$ см⁻³ до $1,76 \times 10^{17}$ см⁻³ легированных

тербием и до $7,2 \times 10^{17}$ см⁻³ легированных тулием. А в образцах, прошедших отжиг в течение 10 часов, наблюдается более малое изменение, с $0,35 \times 10^{17}$ до $0,12 \times 10^{17}$ см⁻³.

Легирование кремния редкоземельными элементами, такими как тулий (Tm) и тербий (Tb), вызывает значительные изменения в его вибрационных свойствах и структуре кристаллической решётки [14-15]. Это обусловлено рассогласованием параметров решётки между атомами Si и Tm (или Tb), что приводит к локальным механическим напряжениям и образованию структурных дефектов в области легирования. Особенно выражен этот эффект в тонкоплёночных и эпитаксиальных структурах, где деформации кристаллической решётки оказывают влияние как на фононные моды,

так и на электронную зонную структуру материала.

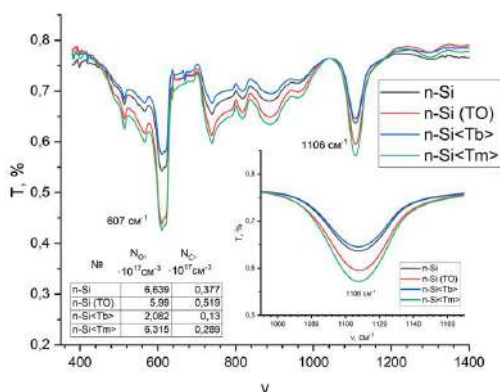


Рис. 1. ИК-спектры образцов кремния марки КЭФ-40

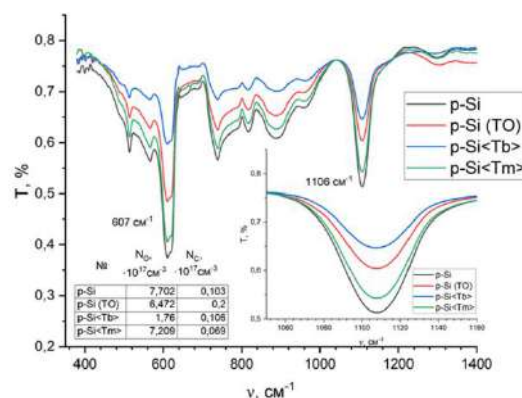


Рис. 2. ИК-спектры образцов кремния марки КДБ-20

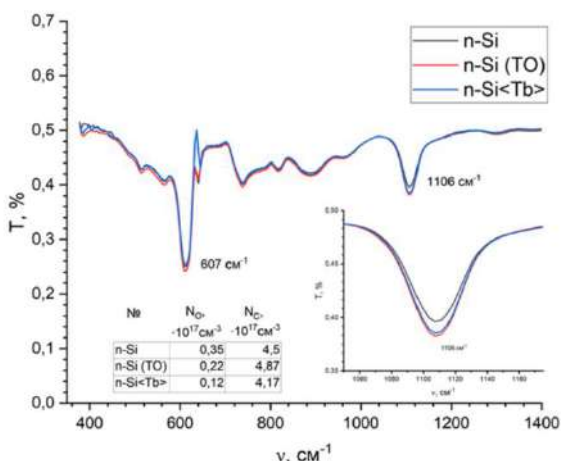


Рис. 3. ИК-спектры образцов кремния марки КЭФ-7,5

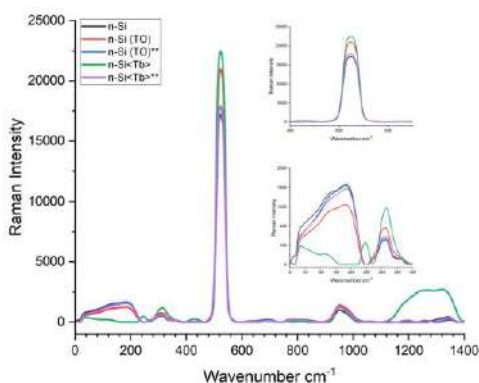


Рис. 4. Рамановские спектры образцов из партии Ф3 КЭФ-7,5, n-Si (TO) и n-Si<Tb> быстрое охлаждение n-Si (TO)** и n-Si<Tb>** медленное охлаждение

В системах твёрдого раствора $\text{Tm}_x\text{Si}_{1-x}$ и $\text{Tb}_x\text{Si}_{1-x}$ значение параметра решётки изменяется в пределах между

соответствующими значениями для кремния и легирующего элемента, что уменьшает степень решёточных искажений.

Однако при увеличении концентрации тулия или тербия и уменьшении толщины слоя возникают дополнительные искажения, напрямую отражающиеся в спектрах комбинационного рассеяния (Раман-спектрах). Для исследования структуры образцов был использован Рамановский спектрометр QERpro+ с длиной волны 785 нм.

При легировании кремния тулием и тербием наблюдается изменение интенсивности фононной моды в области 523 см^{-1} , что, согласно ряду исследований [7-9], указывает на локальные нарушения кристаллической симметрии и перераспределение плотности фононных состояний вблизи центра зоны Бриллюэна. Усиление интенсивности этой моды может быть связано с образованием деформированных областей, а также включением атомов Тм и Tb в кристаллическую решётку кремния, приводящим к локальным изменениям межатомных взаимодействий. Такие эффекты также интерпретируются как проявление сильной связи между редкоземельными атомами и кремниевой матрицей, сопровождающееся модификацией зонной структуры и образованием аморфоподобных включений [13, 15, 17].

Также при легировании тербия при температуре $1250\text{ }^{\circ}\text{C}$ в течение 10 часов с последующим быстрым охлаждением, обнаружен новый пик в области 245 см^{-1} , который соответствует Tb-O. Что может дополнительно подкрепляет, наше предположение об активном взаимодействии редкоземельных элементов с кислородом.

IV. ВЫВОДЫ.

Проведённая экспериментальная работа убедительно демонстрирует, что легирование монокристаллического кремния редкоземельными элементами (тербий и тулий) приводит к значимым изменениям его электрофизических и структурных характеристик. В результате диффузионного отжига кремния марок КЭФ-40, КЭФ-7 и КДБ-20 при температуре $1250\text{ }^{\circ}\text{C}$ с последующим быстрым

охлаждением установлено существенное влияние длительности термообработки и типа редкоземельного элемента на конечные параметры материала.

Экспериментально выявлено, что при легировании кремния марки КЭФ-40 наблюдается уменьшение удельного сопротивления и значительное возрастание концентрации носителей заряда, подтверждающее донорные свойства введённых атомов Tb и Tm. Наиболее выраженный эффект достигнут при использовании тербия, для которого удельное сопротивление снизилось с $41,4$ до $30,3\text{ Ом}\cdot\text{см}$. В случае марки КДБ-20 введение тулия привело не только к снижению удельного сопротивления с $20,3$ до $14,2\text{ Ом}\cdot\text{см}$, но и к кардинальному изменению типа проводимости с дырочного на электронный, что указывает на эффективную компенсацию исходного типа проводимости и формирование n-типа кремния. В образцах марки КЭФ-7 изменения параметров были менее выражены, однако также подтверждают влияние длительности термообработки на степень легирования и формирование дефектных структур.

Нейтронно-активационный анализ позволил количественно подтвердить успешность внедрения атомов Tb и Tm в кремниевую матрицу, с определением точных концентраций на уровне $\sim 10^{16} - 10^{17}\text{ атом/см}^3$. Более того, спектроскопический анализ с помощью Фурье ИК-спектроскопии убедительно доказал эффективность легирования для снижения содержания нежелательных примесей, таких как оптически активный кислород, чья концентрация существенно уменьшилась после введения редкоземельных элементов.

Полученные Раман-спектроскопические данные выявили значительное изменение интенсивности основной фононной моды на 523 см^{-1} , обусловленное локальными структурными искажениями и появлением напряжений в решётке. Дополнительный пик на 245 см^{-1} , идентифицированный как Tb-O, подтвердил химическое взаимодействие

тербия с кислородом и образование дефектных комплексов.

Таким образом, экспериментально доказано, что легирование кремния тербием и тулием является эффективным методом управления его электронными и структурными свойствами, перспективным для улучшения качества полупроводниковых материалов и

дальнейшего использования в области микроэлектроники и полупроводниковых устройств. Представленные результаты формируют основу для дальнейших исследований механизмов взаимодействия редкоземельных элементов с кремнием и оптимизации технологических режимов легирования.

СПИСОК ИСПОЛЬЗОВАННОЙ ЛИТЕРАТУРЫ

- [1]. Utamuradova, S.B., Matchonov, Kh.J., Khamdamov, J.J., & Utemuratova, Kh.J. (2023). Structural Properties of Silicon Doped with Rare Earth Elements Ytterbium. *East European Journal of Physics*, 10(1), 42–47. <https://periodicals.karazin.ua/eejp/article/view/23055>
- [2]. Daliev, K.S., Bahronkulov, Z.E., & Hamdamov, J.J. (2023). Investigation of the Magnetic Properties of Silicon Doped with Rare-Earth Elements. *East European Journal of Physics*, (4), 167–171. <https://doi.org/10.26565/2312-4334-2023-4-18>
- [3]. Nichols A.L. IAEA Co-ordinated Research Project: update of X-ray and gamma-ray decay data standards for detector calibration and other applications. *Appl Radiat Isot.* 2004;60(2-4):247-256. doi: [10.1016/j.apradiso.2003.11.025](https://doi.org/10.1016/j.apradiso.2003.11.025)
- [4]. Sercheli, M.S., & Rettori, C. (2003). Magnetic properties of a-Si films doped with rare-earth elements. *Physical Review B*, 68(17), 174418. <https://doi.org/10.1103/PhysRevB.68.174418>
- [5]. Khuili, M., Fazouan, N., Abou El Makarim, H., Atmani, E.H., Rai, D.P., & Houmad, M. (2020). DFT study of rare earth (Tm, Yb, Ce) doped ZnO: structural, optoelectronic and electrical properties. *arXiv preprint arXiv:2004.10728*. <https://arxiv.org/abs/2004.10728>
- [6]. Yu, Laihao & Zhang, Yingyi & Fu, Tao & Wang, Jie & Cui, Kunkun & Shen, Fuqiang. (2021). Rare Earth Elements Enhanced the Oxidation Resistance of Mo-Si-Based Alloys for High Temperature Application: A Review. *Coatings*. 11. 1144. DOI: [10.3390/COATINGS11091144](https://doi.org/10.3390/COATINGS11091144)
- [7]. Castilho, J.H., Chambouleyron, I., Marques, F.C., Rettori, C., & Alvarez, F. (1991). Electrical conductivity of amorphous silicon doped with rare-earth elements. *Physical Review B*, 43(11), 8946–8953. <https://doi.org/10.1103/PhysRevB.43.8946>
- [8]. T.S.R.C. Murthy, Lucas Reeman, Ji Zou, Vinothini Venkatachalam, Ben Baker, Minshi Wang, Jon Binner, Role of rare earth oxide particles on the oxidation behaviour of silicon carbide coated 2.5D carbon fibre preforms // *Materials Today Communications*, 2020, 24, 100918. DOI: [10.1016/j.mtcomm.2020.100918](https://doi.org/10.1016/j.mtcomm.2020.100918)
- [9]. Hao, S., Oleksak, R.P., Doğan, Ö.N. et al. Insights into oxygen diffusion in rare earth disilicate environmental barrier coatings. *NPJ Comput Mater* 11, 108 (2025).. DOI: [10.1038/s41524-024-01508-y](https://doi.org/10.1038/s41524-024-01508-y)
- [10]. B.E. Egamberdiev, Kh.S. Daliev, I.Kh. Khamidjonov, Sh.B. Norkulov and U.K. Erugliev. On how Doping with Atoms of Gadolinium and Scandium affects the Properties of Silicon // *International Journal of Materials and Nanotechnology*, 2021, 6(2), 45–54. DOI: [10.1234/ijmn.v6i2.206](https://doi.org/10.1234/ijmn.v6i2.206)
- [11]. Wang H., Zhang X., Chen Y. Unexpected doping effects on phonon transport in quasi-one-dimensional van der Waals crystal TiS₃ nanoribbons // *Nature Communications*. – 2023. – Vol. 14, Article 6746. – DOI: [10.1038/s41467-023-41425-0](https://doi.org/10.1038/s41467-023-41425-0)
- [12]. Tian Z., Lee S., Chen G. Effect of lattice mismatch on phonon transmission and interface thermal conductance across dissimilar material interfaces // *Physical Review B*. – 2012. – Vol. 86, No. 5. – P. 054305. – DOI: [10.1103/PhysRevB.86.054305](https://doi.org/10.1103/PhysRevB.86.054305)
- [13]. Zhao H., Xiang H., Wei S.-H. Tuning phonon transport spectrum for better thermoelectric materials // *Science and Technology of Advanced Materials*. – 2019. – Vol. 20, No. 1. – P. 259–270. – DOI: [10.1080/14686996.2018.1548884](https://doi.org/10.1080/14686996.2018.1548884)
- [14]. Gayathri N., Subramanian S., Srinivasan R. Variation of lattice parameters with doping concentration of Tb_{1-x}Y_xMnO₃ and TbMn_{1-x}Ni_xO₃ // *Journal of Physics D: Applied Physics*. – 2013. – Vol. 46. – Article 435302. – DOI: [10.1088/0022-3727/46/43/435302](https://doi.org/10.1088/0022-3727/46/43/435302)
- [15]. Wang H., Zhang X., Chen Y. Unexpected doping effects on phonon transport in quasi-one-dimensional van der Waals crystal TiS₃ nanoribbons // *Nature Communications*. – 2023. – Vol. 14. – Article 6746. – DOI: [10.1038/s41467-023-41425-0](https://doi.org/10.1038/s41467-023-41425-0)
- [16]. Babar S., Weaver J. H. Optical properties and Raman studies of amorphous and disordered silicon structures // *Applied Physics Reviews*, 2015.
- [17]. RRUFF Project Database for Tb₂O₃ reference spectr

MIS BILAN LEGIRLANGAN KREMNIYDA γ -NURLANISH TA'SIRIDA ZARYAD TASHUVCHILARNING REKOMBINATSION XUSUSIYATLARI

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Annotatsiya. Ushbu ishda mis bilan legirlangan p-tip kremniyda γ -nurlanish ta'sirida zaryad tashuvchilarning rekombinatsion xususiyatlari tadqiq etildi. Turli solishtirma qarshilik, Cu konsentratsiyasi, diffuziya harorati va sovitish tezligiga ega namunalar tahlil qilindi. Aniqlanishicha, nurlanish dozasi ortishi bilan zaryad tashuvchilar yashash vaqtining teskari qiymati $1/\tau$ chiziqli qonuniyat asosida ortadi, radiatsion o'zgarish koeffitsienti K_T esa Cu kiritilgan namunalar uchun kamayadi. Bunday holat mis atomlarining kislorod, vakansiya va radiatsion nuqsonlar bilan kompleks markazlar hosil qilishi orqali faol rekombinatsion kanallarni susaytirishi va kremniyning radiatsion barqarorligini oshirishi bilan izohlanadi. Natijalar radiatsion muhitda ishlovchi fotoelektron va mikroelektron qurilmalar uchun material tanlashda hamda texnologik rejimlarni optimallashtirishda ham muhim ahamiyatga ega.

Kalit so'zlar: kremniy, mis, γ -nurlanish, rekombinatsiya, zaryad tashuvchilar yashash vaqti, radiatsion barqarorlik, legirlash, nuqson komplekslari.

Аннотация. В работе исследовано влияние γ -излучения на рекомбинационные свойства носителей заряда в кремнии p-типа, легированном медью. Рассмотрены образцы с различным удельным сопротивлением, концентрацией Cu и скоростью охлаждения после высокотемпературной обработки. Показано, что с увеличением дозы облучения обратное время жизни носителей $1/\tau$ возрастает по линейному закону, а коэффициент радиационного изменения K_T существенно зависит от технологического режима и состава материала. Установлено, что атомы меди, взаимодействуя с кислородом, вакансиями и другими дефектами, могут формировать комплексные центры, уменьшающие концентрацию активных рекомбинационных центров. Полученные результаты подтверждают возможность повышения радиационной стойкости кремния путем контролируемого легирования медью и оптимизации условий термической обработки для радиационных электронных приборов нового поколения.

Ключевые слова: кремний, медь, γ -излучение, рекомбинация, время жизни носителей заряда, радиационная стойкость, легирование, дефектные комплексы.

Annotation. This paper investigates the influence of gamma irradiation on the recombination properties of charge carriers in copper-doped p-type silicon. Samples with different resistivity, copper concentration, diffusion temperature, and cooling rate were analyzed to clarify the role of technological treatment in radiation response. It is shown that, as irradiation dose increases, the inverse carrier lifetime $1/\tau$ follows a linear dependence, while the radiation change coefficient K_T decreases in Cu-containing samples. The observed behavior is attributed to the interaction of copper atoms with oxygen, vacancies, and radiation-induced defects, leading to complex centers that suppress active recombination channels and improve silicon radiation stability.

Keywords: silicon, copper doping, gamma irradiation, recombination, carrier lifetime, radiation stability, defect complexes, p-type silicon.

KIRISH.

Hozirgi kunda kremniy asosidagi yarimo'tkazgich materiallar mikroelektronika, fotoelektronika, energetika va radiatsion muhitda ishlovchi elektron qurilmalar tayyorlashda eng muhim funksional materiallardan biri hisoblanadi. Ularning keng qo'llanilishi, avvalo, elektr va fizik xossalari qulayligi, texnologik jihatdan oson qayta ishlanishi hamda turli tashqi ta'sirlarga nisbatan ma'lum darajada barqarorligi bilan izohlanadi [1]. Ayniqsa, yadroviy energetika, kosmik texnika, yuqori energiyali nurlanish

manbalari bilan bog'liq tizimlar va maxsus elektron asboblarda ishlatiladigan yarimo'tkazgich materiallar uchun radiatsion barqarorlik masalasi alohida ahamiyat kasb etadi [2].

Ma'lumki, γ -nurlanish yarimo'tkazgich kristall panjarasiga ta'sir ko'rsatib, unda turli radiatsion nuqsonlar — vakansiyalar, tugunlararo atomlar, nuqsoniy komplekslar va rekombinatsiya markazlarini hosil qiladi [3]. Bunday nuqsonlar zaryad tashuvchilarning erkin harakatini cheklaydi, ularning konsentratsiyasi, harakatchanligi va ayniqsa yashash vaqtining kamayishiga olib keladi.

Natijada yarimo'tkazgich asboblarning asosiy parametrlari yomonlashadi, ularning ishonchliligi va xizmat muddati pasayadi. Shu sababli nurlanish ta'sirida zaryad tashuvchilarning rekombinatsion xususiyatlarini o'rganish yarimo'tkazgich fizikasining muhim ilmiy va amaliy vazifalaridan biri hisoblanadi.

Kremniyning radiatsion barqarorligini oshirishning samarali yo'llaridan biri uni turli kirishma atomlari bilan legirlashdan iborat. Metall kirishmalar kristall tuzilmada nuqsonlar bilan o'zaro ta'sirlashib, yangi kompleks markazlar hosil qilishi, radiatsiya ta'sirida yuzaga keladigan nuqsonlar konsentratsiyasi va ularning qayta taqsimlanish jarayonlariga ta'sir ko'rsatishi mumkin [4]. Xususan, mis atomlari kremniy kristallida yuqori faollikka ega bo'lib, nuqsonlar bilan murakkab komplekslar hosil qilish xususiyatiga ega [5]. Bu esa γ -nurlanish ta'sirida rekombinatsiya jarayonlarining borishiga, zaryad tashuvchilarning yashash vaqtiga va materialning umumiy elektrofizik xossalriga sezilarli ta'sir ko'rsatishi mumkin [6]. Shu nuqtayi nazardan, ushbu ishning maqsadi mis bilan legirlangan p-tip kremniyda γ -nurlanish ta'sirida zaryad tashuvchilarning yashash vaqti o'zgarishini aniqlash, ushbu o'zgarishlarning rekombinatsion xususiyatini tahlil qilish va jarayonning asosiy fizik mexanizmlarini ochib berishdan iborat.

MATERIALLAR VA METODLAR

Tadqiqotlar p-tipli kremniy monokristall namunalarda olib borildi. Namunalar bor (B) va mis (Cu) atomlari bilan legirlangan bo'lib, ularning elektrofizik xossalari γ -nurlanish ta'sirida o'rganildi. Tadqiq etilgan namunalarning solishtirma elektr qarshiligi $8,8 \div 77 \text{ Om} \cdot \text{sm}$ oralig'ida bo'lgan. Bunday namunalarni tanlashdan maqsad mis kirishmasining radiatsion ta'sir ostida zaryad tashuvchilarning rekombinatsion xususiyatlariga ko'rsatadigan ta'sirini aniqlashdan iborat bo'ldi. Kremniy monokristallarini tayyorlash jarayonida turli texnologik rejimlar, xususan, sovitish tezligi muhim omil sifatida tanlandi. Shu maqsadda namunalarni shakllantirish bosqichida ikki xil sovitish tezligi qo'llanildi: 40 va 250 grad/min. Namunalar γ -nurlanish ta'siriga duchor etildi va nurlanish dozasi ortishi bilan ularning rekombinatsion parametrlari o'zgarishi kuzatildi. Asosiy e'tibor zaryad tashuvchilarning yashash vaqti τ ning o'zgarishiga qaratildi. Tadqiqot jarayonida γ -nurlanish dozasi Φ ortishiga bog'liq holda zaryad tashuvchilar yashash vaqtining teskari qiymati, ya'ni $1/\tau$ kattaligi tahlil qilindi.

NATIJALAR MUHOKAMASI

γ -nurlanish dozasi Φ ortishi bilan mis bilan legirlangan kremniy namunalarda zaryad tashuvchilar yashash vaqtining teskari qiymati, ya'ni $1/\tau$, ma'lum intervallarda chiziqli qonuniyat asosida o'zgaradi. Ushbu bog'lanish quyidagi ifoda bilan tavsiflanadi [7]:

$$\left[\frac{1}{\tau} = \frac{1}{\tau_0} + K_{\tau} \Phi \right] \quad (1)$$

Bu yerda τ_0 — nurlanishdan oldingi zaryad tashuvchilar yashash vaqti; K_{τ} — radiatsion o'zgarish koeffitsienti; Φ — γ -nurlanish dozasi. Mazkur tenglamaga ko'ra, K_{τ} koeffitsienti materialning radiatsiyaga sezgirligini ifodalovchi asosiy parametr hisoblanadi. Uning qiymati qanchalik katta bo'lsa, nurlanish ta'sirida rekombinatsiya jarayonlari shunchalik tez kuchayadi va zaryad tashuvchilar yashash vaqti keskinroq qisqaradi.

2-jadvalda mis konsentratsiyasi ortishi va texnologik ishlov rejimlarining optimallashtirish natijasida K_{τ} qiymatining kamayishi taqdim etilgan. Xususan, oddiy p-Si namunasida $K_{\tau} = 4,8 \times 10^{-12} \text{ sm}^2/\text{s}$ bo'lgan holda, mis bilan legirlangan namunalarda ushbu qiymat sezilarli ravishda pasaygan. Eng kichik qiymat p-Si<P,B,Cu> namunasida kuzatilib, $K_{\tau} = 2,2 \times 10^{-13} \text{ sm}^2/\text{s}$ ni tashkil qilgan. Bu holat mis atomlarining radiatsiya ta'sirida hosil bo'ladigan rekombinatsiya markazlari shakllanishini ma'lum darajada cheklashini va materialning radiatsion barqarorligini oshirishini ko'rsatadi [8]. Shunday qilib, γ -nurlanish ta'sirida zaryad tashuvchilar yashash vaqtining o'zgarish kinetikasi mis kirishmasining konsentratsiyasi va namunalarni tayyorlash texnologiyasiga kuchli bog'liq ekanligi aniqlandi. Ma'lumki, γ -nurlanish ta'sirida kremniy kristallida vakansiyalar V va tugunlararo kremniy atomlari Si_i hosil bo'ladi [9]. Bu radiatsion nuqsonlar kristalldagi kislorod, bor va mis atomlari bilan o'zaro ta'sirlashib, turli kompleks markazlar hosil qiladi. Mis atomlarining muhim xususiyati shundaki, ular kislorod va boshqa nuqsonlar bilan bog'lanib, elektr jihatdan nisbatan nafaol yoki kamroq faol komplekslar shakllantirishi mumkin [10].

$\text{Cu} + \text{O} \rightarrow \text{CuO}, 2\text{Cu} + \text{O} \rightarrow \text{Cu}_2\text{O} \quad (2)$
Bunday komplekslar hosil bo'lishi natijasida kristalldagi A-markazlar (V+O) va boshqa faol rekombinatsiya markazlari konsentratsiyasi kamayishi mumkin [11]. Natijada zaryad tashuvchilar uchun chuqur ushlab qoluvchi markazlar soni qisqaradi va yashash vaqti nisbatan kattaroq qiymatlarda saqlanib qoladi. Shu bilan birga, yuqori haroratli ishlov berish jarayonida mis

atomlarining bir qismi klasterlar yoki silitsidsimon mikrofazalar hosil qilishi ehtimoldan xoli emas.

1-jadval.

Tadqiq etilgan namunalarning asosiy parametrlari

Namuna tipi	ρ , Om·sm	T_{diff} , °C	Sovitish tezligi, grad/min	N_{Cu} , sm ⁻³	τ_0 , c	$K\tau$, sm ² /c
<i>p</i> -Si	10.1	—	250	—	1×10^{-6}	4.8×10^{-12}
<i>p</i> -Si<B,Cu>	9.6	1050	250	2×10^{13}	2×10^{-6}	2.2×10^{-12}
<i>p</i> -Si<B,Cu>	8.8	1150	40	$< 10^{12}$	4×10^{-6}	1.5×10^{-12}
<i>p</i> -Si<P,B,Cu>	77	1250	40	4×10^{14}	9×10^{-6}	2.2×10^{-13}

Nurlanish ta'sirida ushbu tuzilmalarning qayta taqsimlanishi yoki qisman parchalanishi kristallda nuqsonlar muvozanatining o'zgarishiga olib keladi. Bu esa rekombinatsiya markazlarining qayta taqsimlanishiga va zaryad tashuvchilar kinetikasining yangi holatga o'tishiga sabab bo'ladi [12]. Nurlanish dozasi ortishi bilan zaryad tashuvchilarning yashash vaqti bir nechta rekombinatsiya kanallari orqali cheklanadi. Ushbu jarayonni ikki asosiy mexanizm orqali ifodalash mumkin [13]: radiatsiya ta'sirida hosil bo'lgan rekombinatsiya markazlari orqali rekombinatsiya, τ_{RC} ; potensial to'siqlar, mikronuqsonlar yoki fazaviy birjinsli emasliklar bilan bog'liq rekombinatsiya, τ_b . Mazkur holatda umumiy yashash vaqti quyidagi tenglama bilan ifodalanadi [14]:

$$1/\tau = 1/\tau_{RC} + 1/\tau_b \quad (3)$$

Ushbu munosabatdan ko'rinib turibdiki, zaryad tashuvchilar yashash vaqti faqat bitta emas, balki bir vaqtning o'zida bir nechta rekombinatsion kanallar ta'sirida shakllanadi. Ayrim nurlanish dozasi intervallarida tajriba natijalari quyidagi shart bajarilishini ko'rsatdi [15]:

$$1/\tau_{RC} \approx 1/\tau_b \quad (4)$$

Bunday holatda umumiy rekombinatsiya tezligi ma'lum darajada muvozanatlashadi va τ qiymati keskin o'zgarmaydi. Sekin sovitish sharoitida (40 grad/min) kristall panjarasida kirishma atomlari va ichki nuqsonlarning muvozanatliroq taqsimlanishi uchun qulay sharoit yuzaga keladi. Bu esa radiatsiya ta'sirida hosil bo'ladigan yangi nuqsonlarning samarali rekombinatsiya markazlariga aylanish ehtimolini kamaytiradi [16]. Natijada $K\tau$

koeffitsienti kichikroq qiymatga ega bo'ladi va materialning radiatsiyaga sezgirligi pasayadi. Tez sovitishda (250 grad/min) kristalldagi nuqsonlar va kirishma atomlarining nomuvozanatli holati "muzlab qoladi". Bu esa ichki kuchlanishlar, lokal nuqsonlar va metastabil markazlar saqlanib qolishiga sabab bo'ladi [17]. Keyinchalik γ -nurlanish ta'sirida ana shu markazlar qo'shimcha rekombinatsiya kanallari sifatida namoyon bo'lib, zaryad tashuvchilar yashash vaqtining tezroq kamayishiga olib keladi. 2-jadval ma'lumotlari ham shu xulosani tasdiqlaydi. Masalan, 40 grad/min sovitish tezligida tayyorlangan namunalarda $K\tau$ qiymati 250 grad/min sharoitida tayyorlangan namunalarga nisbatan ancha kichik bo'lgan.

XULOSA.

γ -nurlanish ta'sirida kristall tuzilmada rekombinatsiya markazlari konsentratsiyasi oshib borishi bilan zaryad tashuvchilar yashash vaqtining teskari qiymati $1/\tau$ chiziqli ravishda ortishi aniqlandi. Mis bilan legirlangan kremniy namunalari radiatsion o'zgarish koeffitsienti $K\tau$ ning kamayishi kuzatildi. Mis atomlarining kislorod va boshqa nuqsonlar bilan o'zaro ta'sirlashib, turli komplekslar hosil qilishi natijasida radiatsion nuqsonlar konsentratsiyasining kamayishi mumkinligi aniqlandi. Zaryad tashuvchilar yashash vaqti ikkita asosiy rekombinatsiya kanali, ya'ni radiatsion rekombinatsiya markazlari va potensial to'siqlar bilan bog'liq kanallar orqali belgilanishi aniqlandi. Kristallni sovitish tezligi materialning radiatsion barqarorligiga sezilarli ta'sir ko'rsatishi aniqlandi.

ADABIYOTLAR

1. S.M. Sze, K.K. Ng. Physics of Semiconductor Devices. 3rd edition. Wiley-Interscience, Hoboken, New Jersey, 2007, 832 p.
2. M. Moll. Radiation Damage in Silicon Particle Detectors: Microscopic Defects and Macroscopic Properties. PhD Thesis, University of Hamburg, DESY-THESIS-1999-040, 1999, 251 p.
3. G. Lindström. Radiation damage in silicon detectors. Nuclear Instruments and Methods in Physics Research Section A, Vol. 512, №1–2, 2003, pp. 30–43. DOI: 10.1016/S0168-9002(03)01874-6.
4. E.R. Weber. Transition metals in silicon. Applied Physics A, Vol. 30, №1, 1983, pp. 1–22. DOI: 10.1007/BF00617708.
5. A.A. Istratov, E.R. Weber. Physics of Copper in Silicon. Journal of The Electrochemical Society, Vol. 149, №1, 2002, pp. G21–G30. DOI: 10.1149/1.1421348.
6. N.A. Yarykin, J. Weber. Interaction of copper impurity with radiation defects in silicon doped with boron. Semiconductors, Vol. 44, №8, 2010, pp. 983–986. DOI: 10.1134/S1063782610080038.
7. G.C. Messenger, J.P. Spratt. The Effects of Neutron Irradiation on Germanium and Silicon. Proceedings of the IRE, Vol. 46, №6, 1958, pp. 1038–1044.
8. N.A. Yarykin, J. Weber. Interaction of copper impurity with radiation defects in silicon doped with boron. Semiconductors, Vol. 44, №8, 2010, pp. 983–986. DOI: 10.1134/S1063782610080038.
9. G.D. Watkins, J.W. Corbett. Defects in Irradiated Silicon. I. Electron Spin Resonance of the Si-A Center. Physical Review, Vol. 121, №4, 1961, pp. 1001–1016. DOI: 10.1103/PhysRev.121.1001.
10. A.A. Istratov, E.R. Weber. Physics of Copper in Silicon. Journal of The Electrochemical Society, Vol. 149, №1, 2002, pp. G21–G30. DOI: 10.1149/1.1421348.
11. J.L. Lindström. Oxygen-Related Defects in Silicon. MRS Proceedings, Vol. 59, 1985, pp. 45–56. DOI: 10.1557/PROC-59-45.
12. M. Seibt, M. Griess, A.A. Istratov, H. Hedemann, A. Sattler, W. Schröter. Formation and Properties of Copper Silicide Precipitates in Silicon. Physica Status Solidi A, Vol. 166, №1, 1998, pp. 171–182.
13. W. Shockley, W.T. Read. Statistics of the Recombinations of Holes and Electrons. Physical Review, Vol. 87, №5, 1952, pp. 835–842. DOI: 10.1103/PhysRev.87.835.
14. R.N. Hall. Electron-Hole Recombination in Germanium. Physical Review, Vol. 87, №2, 1952, pp. 387–387. DOI: 10.1103/PhysRev.87.387.
15. D.K. Schroder. Semiconductor Material and Device Characterization. 3rd edition. Wiley-IEEE Press, Hoboken, New Jersey, 2006, 800 p.
16. R. Falster, V.V. Voronkov, F. Quast. On the Properties of the Intrinsic Point Defects in Silicon: A Perspective from Crystal Growth and Wafer Processing. Physica Status Solidi B, Vol. 222, №1, 2000, pp. 219–244.
17. G. Lindström. Radiation Damage in Silicon Detectors. Nuclear Instruments and Methods in Physics Research Section A, Vol. 512, №1–2, 2003, pp. 30–43.

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Oltingugurt inson ovqatlanishida muhim element sifatida

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Annotatsiya: Mazkur maqolada oltingugurtning inson organizmidagi biologik va fiziologik ahamiyati, uning oziq-ovqat mahsulotlaridagi manbalari, organizmga tushishi va o'zlashtirilishi, oqsil va aminokislotalar almashinuvidagi roli hamda oltingugurt birikmalarining salomatlikka ta'siri yoritilgan. Oltingugurt inson organizmi uchun zarur bo'lgan makroelementlardan biri bo'lib, asosan metionin va sistein kabi oltingugurt saqlovchi aminokislotalar tarkibida mavjud. Ushbu element oqsillar, fermentlar, kofermentlar va antioksidant tizimlarning normal faoliyatida muhim o'rin tutadi. Maqolada go'sht, baliq, tuxum, sut mahsulotlari, dukkakli o'simliklar, yong'oqlar, piyoz va sarimsoq kabi mahsulotlarning oltingugurt manbai sifatidagi ahamiyati ham tahlil qilinadi. Shuningdek, ratsionda oqsil yetishmovchiligi, oltingugurt almashinuvidagi buzilishlar va oltingugurt birikmalarining ortiqcha iste'moli bilan bog'liq ayrim holatlar haqida ma'lumot beriladi.

Kalit so'zlar: oltingugurt, mineral elementlar, ovqatlanish, aminokislotalar, metionin, sistein, oqsil almashinuvi, glutation, antioksidant, oziq-ovqat mahsulotlari.

Abstract: This article examines the biological and physiological significance of sulfur in the human body, its dietary sources, its intake and absorption, its role in protein and amino acid metabolism, and the effects of sulfur compounds on human health. Sulfur is an essential mineral element for the human body and is primarily present in sulfur-containing amino acids such as methionine and cysteine. This element plays an important role in the normal functioning of proteins, enzymes, coenzymes, and antioxidant systems. The article also analyzes the importance of foods such as meat, fish, eggs, dairy products, legumes, nuts, onions, and garlic as dietary sources of sulfur. Furthermore, it discusses certain conditions associated with dietary protein deficiency, disturbances in sulfur metabolism, and excessive intake of sulfur compounds.

Keywords: sulfur, mineral elements, nutrition, amino acids, methionine, cysteine, protein metabolism, glutathione, antioxidant, food products.

Аннотация: В статье рассматриваются биологическое и физиологическое значение серы в организме человека, её источники в пищевых продуктах, поступление и усвоение организмом, роль в обмене белков и аминокислот, а также влияние соединений серы на состояние здоровья. Сера является одним из необходимых для организма человека минеральных элементов и содержится преимущественно в составе серосодержащих аминокислот, таких как метионин и цистеин. Данный элемент играет важную роль в обеспечении нормального функционирования белков, ферментов, коферментов и антиоксидантных систем. В статье также анализируется значение таких продуктов, как мясо, рыба, яйца, молочные продукты, бобовые, орехи, лук и чеснок, в качестве источников серы. Кроме того, рассматриваются некоторые состояния, связанные с недостаточным поступлением белка с пищей, нарушениями обмена серы и избыточным потреблением соединений серы.

Ключевые слова: сера, минеральные элементы, питание, аминокислоты, метионин, цистеин, белковый обмен, глутатион, антиоксидант, пищевые продукты.

KIRISH

Inson organizmining normal hayot faoliyati ko'plab kimyoviy elementlarning muvozanatli miqdorda mavjud bo'lishiga bog'liq. Mineral moddalar organizmda tuzilishi, katalitik, regulyator va energetik jarayonlarda ishtirok etadi. Ularning ayrimlari juda katta miqdorda talab qilinsa, boshqalari kam miqdorda zarur bo'ladi. Shunga qaramay, har bir elementning organizmdagi o'ziga xos biologik vazifasi mavjud.

Oltingugurt (S) inson organizmi uchun muhim bo'lgan elementlardan biri hisoblanadi. U davriy sistemada VI guruhga mansub bo'lib, atom raqami 16 ga teng. Tabiatda oltingugurt turli mineral va organik birikmalar shaklida

uchraydi. Inson organizmida esa u, avvalo, oltingugurt saqlovchi aminokislotalar — metionin va sistein, shuningdek, boshqa biologik faol birikmalar tarkibida mavjud.

Oltingugurtning oziqlanishdagi ahamiyati ko'pincha uning alohida mineral modda sifatidagi emas, balki biologik molekulalar tarkibidagi o'zni bilan bog'liq holda baholanadi. Ayniqsa, oqsillar sintezi, fermentlarning faoliyati, hujayralarni oksidlovchi stressdan himoya qilish va bir qator metabolik reaksiyalarda oltingugurtning ishtiroki katta ahamiyatga ega.

Zamonaviy ovqatlanish nuqtayi nazaridan oltingugurtning asosiy manbai oqsilga boy oziq-ovqat mahsulotlari

hisoblanadi. Sogʻlom va muvozanatli ratsionda oqsil miqdori yetarli boʻlsa, organizm odatda oltingugurt saqllovchi aminokislotalarni ham yetarli darajada oladi. Shu sababli sogʻlom insonlarda oltingugurtning alohida tanqisligi kam uchraydi.

Mazkur maqolaning maqsadi — oltingugurtning inson ovqatlanishidagi oʻrnini ilmiy jihatdan yoritish, uning organizmdagi biologik vazifalarini, asosiy oziq-ovqat manbalarini va salomatlik uchun ahamiyatini tahlil qilishdan iborat.

1. OLTINGUGURTNING INSON ORGANIZMIDAGI UMUMIY AHAMIYATI

Oltingugurt inson organizmida turli organik birikmalar tarkibida mavjud boʻladi. U organizm massasining nisbatan kichik qismini tashkil etsa-da, koʻplab muhim biologik jarayonlar uchun zarurdir.

Oltingugurtning eng muhim biologik funksiyalaridan biri oqsillar tarkibida ishtirok etishidir. Oqsil molekulalarining tarkibidagi metionin va sistein amino-kislotalari oltingugurt saqlaydi. Sistein molekulalari oʻzaro disulfid bogʻlari hosil qilishi mumkin. Bunday bogʻlar oqsillarning fazoviy tuzilishini barqarorlashtirishda katta ahamiyatga ega.

Masalan, soch, tirnoq va terining asosiy oqsillaridan biri boʻlgan keratin tarkibida sistein koʻp uchraydi. Sistein qoldiqlari oʻrtasida hosil boʻladigan disulfid bogʻlari keratinning mustahkamligi va mexanik xususiyatlarini belgilashda ishtirok etadi. Shu sababli oltingugurt yetarli boʻlishi teri, soch va tirnoqlarning normal holatini taʼminlovchi omillardan biri hisoblanadi.

Oltingugurt, shuningdek, bir qator fermentlar va kofermentlarning tarkibiy qismi sifatida metabolik jarayonlarda qatnashadi. Masalan, koenzim A tarkibidagi tiol guruhi yogʻ kislotalari va uglevodlar almashinuvida muhim reaksiyalarni amalga oshirish imkonini beradi.

Bundan tashqari, oltingugurt hujayralarning antioksidant himoyasida muhim rol oʻynaydi. Sistein aminokislotasi glutation deb ataluvchi muhim antioksidant birikmaning tarkibiga kiradi. Glutination hujayralarni reaktiv kislorod birikmalari va boshqa oksidlovchi

omillarning zararli taʼsiridan himoya qilishda qatnashadi.

2. OLTINGUGURT SAQLOVCHI AMINOKISLOTALARNING AHAMIYATI

Inson ovqatlanishida oltingugurtning asosiy biologik manbai metionin va sistein hisoblanadi.

Metionin inson organizmi uchun almashinmaydigan aminokislota hisoblanadi. Yaʼni organizm uni oʻzida yetarli miqdorda sintez qila olmaydi va u oziq-ovqat orqali olinishi kerak. Metionin oqsil sintezida ishtirok etishidan tashqari, organizmdagi metillanish jarayonlari uchun zarur boʻlgan S-adenozilmationin hosil boʻlishida qatnashadi.

Metionin almashinuvi jarayonida **gomosistein** hosil boʻlishi mumkin. Gomosistein keyinchalik boshqa metabolik yoʻllar orqali qayta metillanishi yoki sistein hosil boʻlishiga olib keluvchi transsulfuratsiya yoʻliga kirishi mumkin. Ushbu jarayonlarda folat, vitamin B12 va vitamin B6 kabi moddalar ham muhim ahamiyatga ega.

Sistein esa shartli almashinuvchi aminokislota boʻlib, organizmda metionindan hosil boʻlishi mumkin. Sistein oqsil sintezida, fermentlar faoliyatida va glutation hosil boʻlishida qatnashadi.

Sisteinning biologik ahamiyatini quyidagicha umumlashtirish mumkin:

1. oqsillar sintezida ishtirok etadi;
2. oqsil molekulalarida disulfid bogʻlarini hosil qiladi;
3. glutation sintezi uchun zarur;
4. oksidlanish-qaytarilish reaksiyalarida qatnashadi;
5. hujayralarning antioksidant himoyasiga hissa qoʻshadi.

Shunday qilib, oltingugurtning ovqatlanishdagi ahamiyatini uning oqsil almashinuvi bilan bevosita bogʻlash mumkin.

3. OLTINGUGURTNING OZIQLANISH MANBALARI

Inson organizmi oltingugurti asosan oziq-ovqat tarkibidagi organik birikmalardan oladi. Ayniqsa, hayvon va oʻsimlik mahsulotlari orasida oqsil miqdori yuqori boʻlgan mahsulotlar oltingugurt saqllovchi aminokislotalarning muhim manbai hisoblanadi.

Goʻsht va baliq mahsulotlari

Mol go'shti, qo'y go'shti, parranda go'shti va baliq tarkibida metionin va sistein mavjud. Ushbu mahsulotlar yuqori biologik qiymatga ega oqsil manbai bo'lgani sababli oltingugurt saqllovchi aminokislotalar bilan ta'minlashda muhim rol o'ynaydi.

Go'sht mahsulotlaridagi oqsil tarkibida oltingugurt saqllovchi aminokislotalar mavjudligi ularning oziqlanish qiymatini oshiradi. Biroq ratsionda faqat hayvon mahsulotlariga tayanish shart emas, chunki o'simlik mahsulotlari ham muhim manba bo'la oladi.

Tuxum

Tuxum, ayniqsa uning oq qismi, yuqori biologik qiymatga ega oqsil saqlaydi. Tuxum oqsillarida sistein va metionin kabi aminokislotalar mavjud. Shu sababli tuxum oltingugurt saqllovchi aminokislotalarning yaxshi manbalaridan biridir.

Sut va sut mahsulotlari

Sut, qatiq, pishloq va boshqa sut mahsulotlari oqsil bilan bir qatorda oltingugurt saqllovchi aminokislotalarni ham ta'minlaydi. Ularning ratsiondagi mavjudligi umumiy oqsil sifatini yaxshilashga xizmat qiladi.

Dukkakli o'simliklar

Loviya, no'xat, yasmiq, soya va boshqa dukkaklilar o'simlik oqsilining muhim manbalaridir. Ularning tarkibida oltingugurt saqllovchi aminokislotalar mavjud bo'lsa-da, ayrim dukkakli mahsulotlarda metionin miqdori hayvon oqsillariga nisbatan pastroq bo'lishi mumkin.

Shu sababli turli xil o'simlik oqsillarini birgalikda iste'mol qilish oqsilning aminokislota tarkibini muvozanatlashtirishga yordam beradi.

Yong'oq va urug'lar

Yong'oq, bodom, kungaboqar urug'i, kunjut va boshqa urug'lar ham oqsil hamda biologik faol moddalar manbai sifatida muhimdir. Ularning tarkibidagi oltingugurt saqllovchi aminokislotalar umumiy ratsionga qo'shimcha hissa qo'shadi.

Piyoza va sarimsoq

Piyoza va sarimsoq tarkibida oqsillardan tashqari, oltingugurt saqllovchi o'ziga xos organik birikmalar mavjud. Ular tarkibidagi allitsin va boshqa oltingugurtli birikmalar ushbu sabzavotlarning o'ziga xos hidi va ta'mini shakllantiradi.

Sarimsoq va piyoza tarkibidagi oltingugurt birikmalari biologik faolligi bilan ham qiziqish uyg'otadi. Ularning antioksidant va boshqa fiziologik ta'sirlari bo'yicha ko'plab ilmiy tadqiqotlar olib borilgan.

4. OLTINGUGURT VA OQSIL ALMASHINUVI

Oltingugurtning ovqatlanishdagi eng muhim jihatlaridan biri uning oqsil almashinuvi bilan bog'liqligidir. Inson organizmi oqsillardan yangi hujayralar, fermentlar, gormonlar va boshqa biologik molekulalarni sintez qilish uchun foydalanadi.

Oqsillar tarkibidagi metionin va sistein oltingugurt almashinuvining markaziy komponentlari hisoblanadi. Metionin almashinuvi orqali hosil bo'ladigan birikmalar ko'plab metabolik reaksiyalarda ishtirok etadi. Oltingugurt saqllovchi aminokislotalarning yetarli miqdorda qabul qilinishi:

- yangi oqsillar sintezini;
 - to'qimalarning yangilanishini;
 - fermentlar hosil bo'lishini;
 - antioksidant himoyani;
 - normal hujayra metabolizmini
- qo'llab-quvvatlashga yordam beradi.

Oziq-ovqat tarkibidagi oqsil miqdori juda past bo'lgan ratsionlarda nafaqat umumiy oqsil, balki oltingugurt saqllovchi aminokislotalarning ham yetishmovchiligi yuzaga kelishi mumkin. Bunday holat uzoq muddat davom etsa, organizmning tiklanish jarayonlari, mushak oqsillari sintezi va boshqa fiziologik funksiyalariga salbiy ta'sir ko'rsatishi mumkin.

Shu sababli sog'lom ovqatlanish tamoyillaridan biri turli manbalardan sifatli oqsil olishdan iborat.

5. OLTINGUGURT VA ANTIOKSIDANT HIMOYA

Inson organizmida doimiy ravishda oksidlanish jarayonlari sodir bo'ladi. Ushbu jarayonlarda reaktiv kislorod birikmalari hosil bo'lishi mumkin. Ularning haddan tashqari ko'payishi oksidlovchi stressga olib kelishi ehtimoli mavjud.

Organizm oksidlovchi stressga qarshi turish uchun murakkab antioksidant tizimga ega. Ushbu tizimning muhim komponentlaridan biri **glutation** hisoblanadi. Glutation uchta aminokislotalardan tashkil topgan bo'lib, uning tarkibiga sistein kiradi.

Sisteinning tiol guruhi glutationning antioksidant faoliyatida muhim ahamiyatga ega.

Glutation:

- hujayralarni oksidlovchi stressdan himoya qilishda;
- ayrim zararli birikmalarni zararsizlantirishda;
- hujayraning oksidlanish-qaytarilish muvozanatini saqlashda;
- boshqa antioksidant tizimlar faoliyatini qo'llab-quvvatlashda ishtirok etadi.

Shuning uchun ratsionda sistein va metioninning yetarli bo'lishi organizmning normal antioksidant himoyasini ta'minlashda muhim omillardan biri hisoblanadi.

6. OLTINGUGURTNING SOCH, TERI VA TIRNOQLAR UCHUN AHAMIYATI

Teri, soch va tirnoqlarning asosiy strukturaviy oqsillaridan biri keratin hisoblanadi. Keratin tarkibida sistein ko'p bo'lib, sistein qoldiqlari orasida disulfid bog'lari hosil bo'ladi.

Disulfid bog'lari keratin molekularining mustahkamligini oshiradi. Shu sababli oltingugurt saqlovchi aminokislotalar teri, soch va tirnoqlarning strukturaviy yaxlitligini ta'minlashda muhim ahamiyatga ega.

Sochning mustahkamligi va elastikligi ko'plab omillarga bog'liq bo'lsa-da, oqsil va oltingugurt saqlovchi aminokislotalarning yetarli darajada iste'mol qilinishi organizmning normal keratin sintezini qo'llab-quvvatlaydi.

Shuni ta'kidlash kerakki, soch yoki tirnoq bilan bog'liq muammolarni faqat oltingugurt yetishmasligi bilan izohlash to'g'ri emas. Bunday holatlarga temir, rux, oqsil, vitaminlar yetishmovchiligi, gormonal o'zgarishlar va boshqa ko'plab omillar sabab bo'lishi mumkin.

7. OLTINGUGURT YETISHMOVCHILIGI VA ORTIQCHALIGI

Sog'lom insonlarda oltingugurtning alohida yetishmovchiligi juda kam uchraydi. Buning asosiy sababi — oltingugurt saqlovchi aminokislotalarning ko'plab odatiy oziq-ovqat mahsulotlarida mavjudligidir.

Odatda muvozanatli ratsionda go'sht, baliq, tuxum, sut mahsulotlari yoki o'simlik

oqsillarining yetarli miqdorda bo'lishi organizmning oltingugurtga bo'lgan ehtiyojini qoplaydi.

Biroq og'ir oqsil yetishmovchiligi, uzoq muddatli noto'g'ri ovqatlanish yoki ayrim patologik holatlarda oltingugurt saqlovchi aminokislotalarning yetarli qabul qilinmasligi kuzatilishi mumkin.

Oltingugurtning o'zi bilan bog'liq ortiqcha iste'mol masalasi esa oziq-ovqat tarkibidagi shakliga bog'liq. Masalan, oziq-ovqat sanoatida ayrim oltingugurt birikmalari konservant sifatida ishlatiladi. Sulfitlarga sezgir bo'lgan ayrim odamlarda bunday moddalar noxush reaksiyalarni keltirib chiqarishi mumkin.

Shu sababli oltingugurtning foydali biologik funksiyalari bilan oziq-ovqat qo'shimchalari sifatida ishlatiladigan ayrim oltingugurt birikmalarini bir-biridan farqlash zarur.

8. OLTINGUGURTGA BOY MAHSULOTLARDAN FOYDALANISHNING OVQATLANISH NUQTAI NAZARIDAN AHAMIYATI

Sog'lom ovqatlanishda asosiy maqsad ma'lum bir mineral elementni alohida ko'p miqdorda iste'mol qilish emas, balki barcha zarur oziq moddalarni muvozanatli tarzda qabul qilishdir.

Oltingugurt uchun ham xuddi shu tamoyil amal qiladi. Ratsion quyidagi mahsulot guruhlarini o'z ichiga olishi maqsadga muvofiq:

- go'sht yoki boshqa sifatli oqsil manbalari;
- baliq va tuxum;
- sut va sut mahsulotlari;
- dukkakli o'simliklar;
- yong'oq va urug'lar;
- turli sabzavot va mevalar;
- don mahsulotlari.

O'simlikka asoslangan ratsionda turli oqsil manbalarini kombinatsiyalash orqali aminokislotalar tarkibini yaxshilash mumkin. Masalan, dukkaklilarni don mahsulotlari bilan birgalikda iste'mol qilish oqsil sifatini yaxshilashga yordam beradi. Bundan tashqari, ratsionning xilma-xilligi organizmni nafaqat oltingugurt, balki temir, rux, magniy, kalsiy, vitaminlar, tola va boshqa biologik faol

moddalar bilan ham ta'minlash imkonini beradi.

9. OLTINGUGURTNING INSON SALOMATLIGIDAGI ISTIQBOLLI AHAMIYATI

So'nggi yillarda oltingugurt saqllovchi biologik faol birikmalarga bo'lgan ilmiy qiziqish ortib bormoqda. Ayniqsa, sarimsoq, piyoz, karamdoshlarga mansub sabzavotlar tarkibidagi oltingugurtli birikmalarning biologik xususiyatlari tadqiq qilinmoqda.

Bunday birikmalar antioksidant, hujayra signalizatsiyasi va boshqa fiziologik jarayonlarda ishtirok etishi mumkin. Biroq oziq-ovqat tarkibidagi tabiiy oltingugurt birikmalarining inson salomatligiga ta'sirini baholashda ularning turi, miqdori, biologik o'zlashtirilishi va individual xususiyatlarni hisobga olish zarur.

Shuningdek, oltingugurt almashinuvi bilan bog'liq metabolik jarayonlar yurak-qon tomir, asab va boshqa tizimlar faoliyati bilan ham bevosita yoki bilvosita bog'liq bo'lishi mumkin. Masalan, metionin almashinuvida hosil bo'ladigan gomosistein miqdorining ortishi bir qator metabolik va fiziologik omillar bilan bog'liq. Shu sababli oltingugurt almashinuvini faqat bitta element nuqtayi nazaridan emas, balki folat, B6 va B12 vitaminlari hamda umumiy oqsil almashinuvi bilan birgalikda o'rganish muhimdir.

XULOSA

Oltingugurt inson organizmi uchun muhim biologik elementlardan biri bo'lib, uning asosiy ahamiyati oltingugurt saqllovchi aminokislotalar va boshqa organik birikmalar tarkibida namoyon bo'ladi. Metionin va sistein oqsil sintezi, hujayralarning normal faoliyati, fermentlar faoliyati va antioksidant himoya jarayonlarida muhim o'rin tutadi.

Oltingugurt, ayniqsa, keratin tarkibida mavjud bo'lganligi sababli teri, soch va tirnoqlarning strukturaviy xususiyatlarini saqlashda ishtirok etadi. Sisteinning glutation tarkibiga kirishi esa oltingugurtning hujayralarni oksidlovchi stressdan himoya qilishdagi ahamiyatini ko'rsatadi.

Inson oltingugurtini asosan go'sht, baliq, tuxum, sut mahsulotlari, dukkaklilar,

yong'oqlar va boshqa oqsilga boy mahsulotlar orqali oladi. Piyoz va sarimsoq kabi mahsulotlar esa o'ziga xos oltingugurt saqllovchi biologik faol birikmalarning manbai hisoblanadi.

Muvozanatli va xilma-xil ovqatlanadigan sog'lom insonlarda oltingugurt yetishmovchiligi odatda alohida muammo sifatida yuzaga kelmaydi. Shuning uchun oltingugurtni qo'shimcha preparatlar shaklida qabul qilishdan ko'ra, uni tabiiy va sifatli oziq-ovqat mahsulotlari orqali olish maqsadga muvofiqdir.

Umuman olganda, oltingugurtning inson ovqatlanishidagi ahamiyati uning oqsil almashinuvi, fermentativ jarayonlar, antioksidant himoya va to'qimalarning strukturaviy yaxlitligini ta'minlashdagi ishtiroki bilan belgilanadi. Kelgusida oltingugurt saqllovchi oziq-ovqat birikmalarining metabolik jarayonlar va inson salomatligiga ta'sirini yanada chuqur o'rganish oziq-ovqat biokimyosi va profilaktik ovqatlanish sohalari uchun muhim ilmiy yo'nalishlardan biri bo'lib qoladi.

FOYDALANILGAN ADABIYOTLAR

- [1]. Gropper S. S., Smith J. L., Carr T. P. Advanced Nutrition and Human Metabolism. Cengage Learning.
- [2]. Ross A. C., Caballero B., Cousins R. J., Tucker K. L., Ziegler T. R. Modern Nutrition in Health and Disease. Wolters Kluwer.
- [3]. Murray R. K., Bender D. A., Botham K. M. va boshqalar. Harper's Illustrated Biochemistry. McGraw-Hill.
- [4]. Nelson D. L., Cox M. M. Lehninger Principles of Biochemistry. W. H. Freeman.
- [5]. Gropper S. S., Smith J. L. Advanced Nutrition and Human Metabolism. Cengage Learning.
- [6]. World Health Organization. Healthy Diet. Geneva: WHO.
- [7]. Institute of Medicine. Dietary Reference Intakes for Energy, Carbohydrate, Fiber, Fat, Fatty Acids, Cholesterol, Protein and Amino Acids. National Academies Press.
- [8]. National Institutes of Health, Office of Dietary Supplements. Dietary Supplements and Nutrient Information.
- [9]. Berg J. M., Tymoczko J. L., Stryer L. Biochemistry. W. H. Freeman.
- [10]. Devlin T. M. Textbook of Biochemistry with Clinical Correlations. Wiley.

Effect of High External Pressure on the Structure and Dissociation of Nickel-Silicide Precipitates in Single-Crystal Silicon

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Abstract. This work investigates the effect of high external hydrostatic pressure on the structure and stability of nickel-silicide precipitates in single-crystal silicon. It is shown that external hydrostatic pressure can compensate the internal elastic stresses that arise from the lattice mismatch between the silicon matrix and the silicide inclusions. At a critical pressure, $P_{cr} \approx \sigma_{NiSi_2}$, the onset of precipitate dissociation is observed, corresponding to a transition of the system toward a thermodynamically more stable state.

Electron-microscopy analysis established that the precipitates possess a layered structure comprising the Ni_2Si , $NiSi$, $NiSi_2$, and oxide phases, distinguished by contrast and by silicon content across the layers. The kinetics of decomposition were analyzed with reference to diffusion processes and the role of vacancies. The theoretical description is based on the Mott–Nabarro model, which relates internal stresses to baric effects and to the conditions for phase instability.

The results confirm that external pressure can initiate the decomposition of silicide inclusions and also influences the activation energy of diffusion. These findings are relevant to understanding the mechanisms of stability and breakdown of metallic and silicide phases in silicon, which is important for improving doping technologies and contact formation in microelectronic structures.

Аннотация. В работе исследуется влияние высокого внешнего гидростатического давления на структуру и стабильность выделений силицида никеля в монокристаллическом кремнии. Показано, что внешнее гидростатическое давление способно компенсировать внутренние упругие напряжения, возникающие из-за несоответствия параметров решетки между кремниевой матрицей и силицидными включениями. При критическом давлении $P_{cr} \approx \sigma_{NiSi_2}$ наблюдается начало диссоциации преципитатов, соответствующее переходу системы в термодинамически более стабильное состояние.

Электронно-микроскопический анализ установил, что преципитаты обладают слоистой структурой, включающей фазы Ni_2Si , $NiSi$, $NiSi_2$ и оксидные фазы, различающиеся по контрасту и содержанию кремния по слоям. Кинетика распада проанализирована с учетом диффузионных процессов и роли вакансий. Теоретическое описание основано на модели Мотта — Набарро, связывающей внутренние напряжения с барическими эффектами и условиями фазовой нестабильности.

Результаты подтверждают, что внешнее давление может инициировать распад силицидных включений, а также влияет на энергию активации диффузии. Эти данные важны для понимания механизмов стабильности и разрушения металлических и силицидных фаз в кремнии, что имеет важное значение для совершенствования технологий легирования и формирования контактов в микроэлектронных структурах.

Annotatsiya. Ushbu ishda yuqori tashqi gidrostatik bosimning monokristalli kremniy tarkibidagi nikel silisid cho'kmalarining tuzilishi va turg'unligiga ta'siri o'rganiladi. Tashqi gidrostatik bosim kremniy matritsasi va silisid qo'shilmalari orasidagi panjara mos kelmasligidan kelib chiqadigan ichki elastik kuchlanishlarni kompensatsiya qilishga qodir ekanligi ko'rsatilgan. Kritik bosimda $P_{cr} \approx \sigma_{NiSi_2}$ pretsipitatlarning dissotsiatsiyalanishi boshlanishi kuzatiladi, bu tizimning termodinamik jihatdan barqarorroq holatga o'tishiga mos keladi.

Elektron-mikroskopik tahlil pretsipitatlar Ni_2Si , $NiSi$, $NiSi_2$ va oksid fazalarini o'z ichiga olgan qatlamli tuzilishga ega ekanligini, ular qatlamlar bo'ylab kontrast va kremniy miqdori bilan farqlanishini aniqladi. Parchalanish kinetikasi diffuziya jarayonlari va vakansiyalarning roli bilan bog'liq holda tahlil qilindi. Nazariy tavsif ichki kuchlanishlarni barik effektlar va faza turg'unligi shartlari bilan bog'laydigan Mott–Nabarro modeliga asoslangan.

Olingan natijalar tashqi bosim silisid qo'shilmalarining parchalanishini boshlashi hamda diffuziyaning faollashish energiyasiga ta'sir qilishini tasdiqlaydi. Ushbu ma'lumotlar kremniydagi metall va silisid fazalarining turg'unligi va yemirilish mexanizmlarini tushunish uchun muhim bo'lib, mikroelektron tuzilmalarda legirlash texnologiyalari va kontaktlarni shakllantirishni takomillashtirishda ahamiyatlidir.

Keywords: nickel silicides, precipitates, high pressure, dissociation, internal stresses, diffusion.

1. Introduction

The formation and decomposition of metal and silicide precipitates in silicon is an important topic in semiconductor physics, since such objects affect the electrical and structural properties of the material. In particular, nickel precipitates — in the form of various silicides (Ni_2Si , NiSi , NiSi_2 , and others) — can exist within the silicon matrix in a thermodynamically metastable state, held there by internal elastic stresses arising from the lattice mismatch between the host matrix and the inclusion. External hydrostatic pressure can “compensate” these stresses, leading to precipitate dissociation and a transition of the system to a more stable state.

The aim of this work is to investigate the effect of high external pressure on the structure and stability of nickel-silicide precipitates, to identify the conditions for the onset of their decomposition, and to propose a structural model of the precipitate that accounts for its layered structure and internal stresses.

The literature highlights the following key points: internal elastic stresses in precipitates are commonly estimated using theoretical approaches such as the Mott–Nabarro theory [3]; precipitates in Si:Ni frequently consist of several nickel-silicide phases together with oxides (Ni_2Si , NiSi , NiSi_2 , oxides); Ref. [1] considers baric effects on the decomposition of inclusions in silicon; and recent studies report direct observation of silicide formation in nanostructures, e.g. in Si nanowires [2].

Accordingly, the objectives of the present work are: (1) to experimentally determine the critical external pressure P at which precipitate dissociation begins; (2) to compare this pressure with the theoretical internal stress σ of the outer precipitate layer (e.g., NiSi_2); (3) to propose a structural model of the precipitate layer, taking into account the layered, contrast-differentiated structure observed in electron-microscopy images; and (4) to analyze the dissociation kinetics with reference to vacancies and diffusion mechanisms.

2. Theoretical background

2.1. Internal elastic stresses and the Mott–Nabarro theory

When a precipitate whose lattice parameter differs from that of the host matrix is embedded within a crystal, a lattice mismatch (misfit strain)

arises. This produces internal elastic stresses that can hold the precipitate in a state removed from thermodynamic equilibrium.

The Mott–Nabarro theory [3] provides a framework for estimating the critical stress at which dislocations or other defects begin to overcome this barrier, allowing the inclusion to decompose or transform.

In a simple approximation, an externally applied pressure P acting against the internal stress σ of the precipitate's outer layer can be taken to “compensate” σ . When $P \geq \sigma$, the precipitate becomes unstable and begins to dissociate. Note that this equality treats σ as an effectively hydrostatic (volumetric) quantity; misfit-strain fields in general also contain a deviatoric (shear) component that a purely hydrostatic external pressure does not directly oppose, so $P \approx \sigma$ should be understood as a first-order estimate rather than an exact mechanical balance — a point worth stating explicitly if this derivation is to be defended in review.

2.2. Dissociation kinetics and diffusion limitations

Even when the external pressure is sufficient to overcome the mechanical barrier, the dissociation process itself requires the migration of atoms and defects. In the vicinity of the precipitate there is reported to be an elevated vacancy concentration, which allows atoms detaching from the precipitate to diffuse over a characteristic distance $l \sim 10^{-8}$ cm within a time t , according to

$$l = \sqrt{(Dt)} \quad (1)$$

where D is the diffusion coefficient. Even for small D , this diffusion can be sufficient to produce dissociation, provided that t is sufficiently long.

A further possible contribution is a baric “quenching” effect, analogous to thermal quenching: pressure may alter — in general, raise — the activation energy of diffusion, thereby slowing the process.

2.3. Nickel-silicide phases and their properties

Nickel forms several well-characterized silicide phases — Ni_2Si , NiSi , and NiSi_2 — that differ in lattice parameter, density, and thermal and mechanical properties. The outer layer of the precipitate is commonly modeled as NiSi_2 , and it is the stress of this layer that is most often

assumed to be compensated by the applied external pressure P .

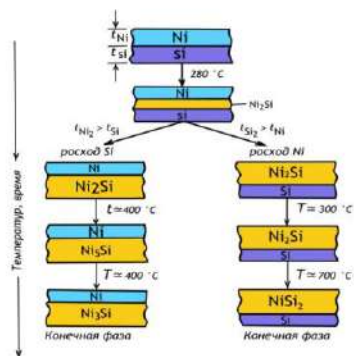


Figure 1. Schematic mechanism of nickel-silicide phase formation in the silicon matrix.

The formation of silicide phases during thermal treatment of silicon with a thin nickel overlayer has also been studied in the context of nickel-silicide-mediated crystallization of amorphous silicon [4,5].

Studies of baric effects in silicon further indicate that external hydrostatic loading can promote the dissociation of inclusions once the internal stresses are already close to the stability limit [1].

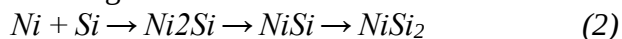
As illustrated in Fig. 1, a thin nickel layer is initially deposited on a silicon substrate. Upon heating to approximately 280 °C, interdiffusion of Ni and Si atoms begins at the interface, giving rise to the primary Ni_2Si phase [4] (the source manuscript cites ref. [3] here — the Mott–Nabarro dislocation-theory monograph — for this silicide-formation-temperature claim; that citation appears to be misplaced and should instead point to a silicide-kinetics reference such as [4] or [5], as corrected here).

The subsequent evolution of the reaction is governed by which reactant is locally depleted first:

(i) When silicon is preferentially consumed, Si atoms diffuse into the nickel layer, forming Ni_2Si . Upon further heating to ≈ 400 °C, the source diagram indicates a transition through an intermediate phase and on to a final phase labeled “ Ni_3Si ” — as noted above, this branch and its intermediate phase require verification against the phase diagram before being stated as an established result.

(ii) When nickel is preferentially consumed, Ni atoms diffuse into the silicon substrate, first forming Ni_2Si at $T \approx 300$ °C, and then, upon heating to ≈ 700 °C, the NiSi_2 phase — the thermodynamically stable end phase of the Ni–Si system.

In general, the process is characterized by the sequential formation of silicide phases according to the scheme:



The phase evolution depends on the annealing temperature and time, and on the relative interdiffusion rates of the two components. At high temperature, NiSi_2 is the equilibrium phase, possessing the lowest Gibbs energy and remaining stable under further thermal treatment.

3. Experimental

3.1. Sample preparation

Samples were fabricated from n-type single-crystal silicon with a resistivity of 15–20 $\Omega \cdot \text{cm}$. The surface was subjected to mechanical and chemical cleaning, after which a metallic nickel layer of nominal thickness $d = 45$ nm was deposited.

Nickel was diffused into the silicon in a horizontal resistance furnace (SOUL-4) over the temperature range $T = 1000$ – 1550 K. The source manuscript states only that a series of experiments was “almost certainly” carried out at various temperatures and pressures — language that cannot be carried into a Methods section. Please confirm and supply: the actual applied hydrostatic pressure range and the number of discrete pressure steps used, the pressure apparatus (e.g., piston-cylinder or diamond-anvil cell), the loading duration/protocol, and whether pressure and temperature were varied independently or jointly. Once provided, this paragraph can be written with the precision a Methods section requires.

After diffusion, the thickness of the Ni layer was measured, and — per the source manuscript, which states this only as a possibility — the precipitate structure was examined by electron microscopy or other methods. Please confirm which structural characterization technique(s) were actually applied to the diffused (and, separately, to the pressure-treated) samples, allowing the appearance of different silicide phases to be recorded [1].

3.2. Electron microscopy and structural analysis

Electron-microscopy examination showed that the precipitates are not homogeneous inclusions of uniform structure: they consist of several layers with differing contrast. The central layer (core) exhibits the brightest contrast. Micrographs allow a clear boundary to be distinguished between the precipitate and the silicon matrix, and an intermediate layer is occasionally observed between the background and the high-contrast core — evidence of an elevated vacancy concentration in the near-precipitate region.

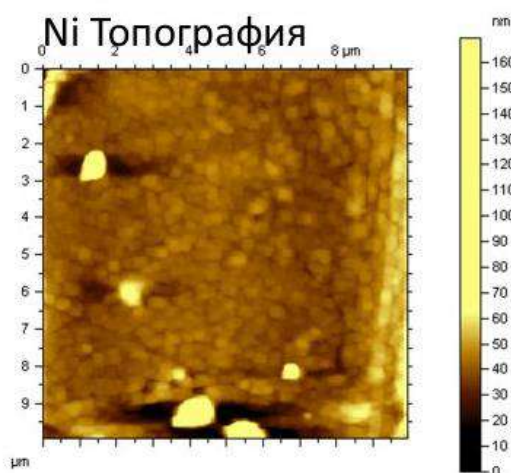


Figure 2. AFM surface topography of the nickel-layer sample following exposure to high pressure [condition to be confirmed].

Fig. 2 shows an atomic force microscopy (AFM) topographic map of the nickel-layer surface. The color scale (right) indicates the distribution of surface heights over the range 0 to ≈ 160 nm.

The surface exhibits a pronounced granular structure, with isolated raised features (islands) reaching heights of 150–160 nm. Such features correspond to regions of locally elevated nickel concentration and may be associated with aggregation during deposition or with subsequent thermal annealing.

The average surface roughness (R_a) is on the order of several tens of nanometers, indicating morphological inhomogeneity and a possible dependence of the diffusion process on local topography.

Taking into account the differing densities of the various phases and the observed contrast, a structural model was proposed in which the

precipitate core is predominantly Ni, surrounded by layers of increasing Si content that form the nickel-silicide layers, up to the transition to the surrounding Si crystal.

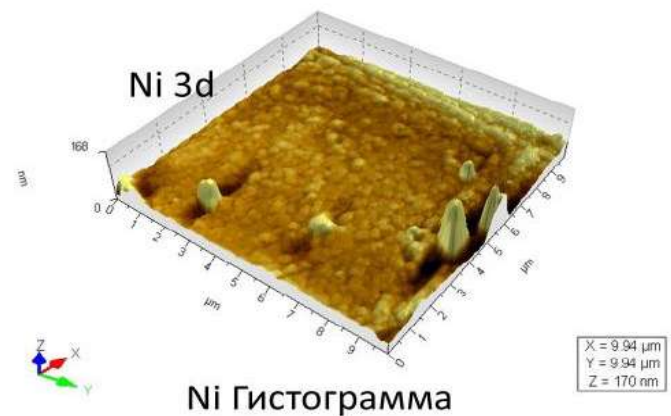


Figure 3. Three-dimensional AFM topographic histogram of the nickel-layer sample following exposure to high pressure [condition to be confirmed].

Fig. 3 presents a three-dimensional AFM topographic histogram of the nickel-layer surface. The scanned area is approximately $9.94 \times 9.94 \mu\text{m}^2$, and the maximum relief height reaches ≈ 170 nm.

The surface again shows a pronounced granular structure, with isolated raised features — local nickel aggregates — whose height substantially exceeds the average surface level. Such irregularities may result from non-uniform crystallite growth during nickel deposition or its subsequent diffusion during thermal treatment.

The morphological analysis indicates structural inhomogeneity of the nickel coating and a possible influence of these micro-scale irregularities on diffusion, silicide formation, and the mechanical stability of the system under high pressure [2].

3.3. Determination of the critical dissociation pressure

In the experiments, external hydrostatic pressure P was applied incrementally, and the value at which precipitate decomposition began to be observed — structural change, size reduction, loss of contrast — was recorded. This value, P_{cr} , is taken to correspond to the internal stress σ of the outer precipitate layer, in the present case the NiSi_2 layer [4]. No numerical value of P_{cr} — nor of the applied pressure range or step size — is given anywhere in the source manuscript. Please supply the measured value(s)

so that the central quantitative claim of this paper ($P_{cr} \approx \sigma$) can be stated explicitly rather than only qualitatively.

$$P_{cr} \approx \sigma_{NiSi_2}(\text{layer})$$

4. Results

4.1. Composition and phases of the precipitates

Structural analysis showed that in the Si<Ni> samples, precipitates predominantly consist of the Ni_2Si , $NiSi$, $NiSi_2$, and silicide-oxide phases (e.g., $Ni_2Si_2(OH)_4$). Phase formation begins at comparatively low temperatures, with the higher-melting-point — i.e., more thermally stable — phases forming first.

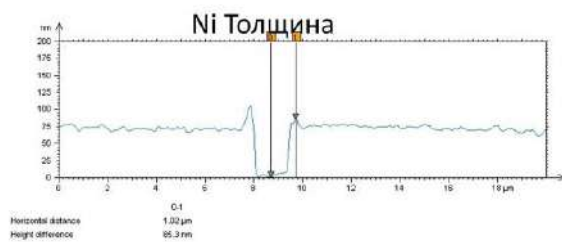


Figure 4. AFM line profile across the nickel layer grown by the diffusion method on the surface of an n-Si single crystal (measured step height: 85.3 nm).

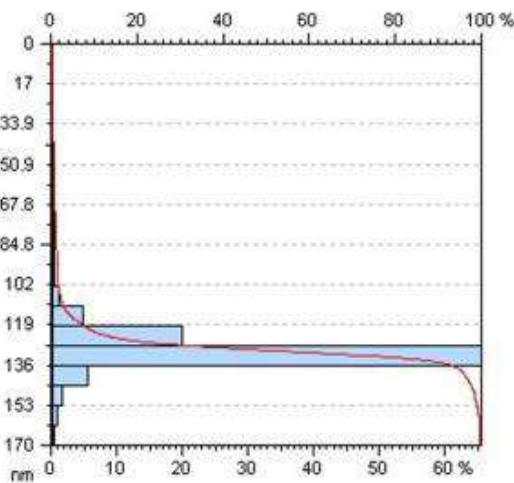


Figure 5. Distribution of the most frequently occurring nickel-film surface heights (AFM height histogram).

The thickness d of the metallic layer strongly influences the resulting phase composition; at the nominal thickness of $d = 45$ nm, the silicide phases were found to be most sensitive to the applied external pressure. Three different values are given in the source material for what is nominally the same quantity (“Ni layer thickness”): $d = 45$ nm (stated as the

optimal deposition thickness, §3.1), 85.3 nm (the step height measured from the AFM line profile, Fig. 4), and ≈ 136 –153 nm (the modal range of the AFM surface-height distribution, Fig. 5). Please clarify whether these refer to different quantities (as-deposited film thickness vs. measured step height vs. surface-roughness statistics) or to different samples, so the values can be reconciled.

4.2. Critical pressure and correspondence with internal stress

The experimentally determined value of P_{cr} was found to agree in order of magnitude with the stress σ corresponding to the outer precipitate layer ($NiSi_2$), supporting a model in which the outer phase plays the leading role in maintaining the precipitate's metastable structure [5]. As noted in §3.3, the underlying numerical values of P_{cr} and σ are not reported in the source manuscript; this order-of-magnitude agreement should be substantiated with explicit figures (e.g., in a table) before submission.

For $P \geq P_{cr}$, dissociation of the inclusions is observed, and the Si + precipitate solid solution tends toward a more stable equilibrium state.

4.3. Dissociation kinetics

From the estimated diffusion length $l = \sqrt{(Dt)}$, it follows that atoms detaching from the precipitate can migrate an appreciable distance even for small D , provided that t is sufficiently long. The presence of vacancies enhances this effect [6].

A possible contribution from baric “quenching,” which may raise the activation energy of diffusion under pressure, cannot be excluded; a complete kinetic model will require further experimental investigation of this effect.

5. Discussion

5.1. Comparison with the literature

The work of Fistul et al. on baric effects in silicon already proposed that external pressure can promote the decomposition of inclusions once the internal stresses are close to the compensation limit [1]. The present results are consistent with this theoretical premise.

Recent in situ TEM studies of silicide formation in nanostructures reveal the dynamics of phase formation and the role of defects in these processes [2]; in particular, defects forming near silicide boundaries can influence diffusion pathways and phase stability.

Studies of amorphous-silicon crystallization mediated by nickel silicides further indicate that NiSi_2 can migrate through the crystalline Si matrix, influencing the growth and structure of precipitates [4,5].

5.2. Limitations and assumptions

The model assumes that the outer precipitate layer is specifically NiSi_2 , which may be a simplification; in reality, a compositional gradient may be present.

The contribution of the baric effect to the diffusion coefficient has not been quantified.

The experimental value(s) of P_{cr} (not yet numerically reported — see §3.3) require confirmation through replicate measurements.

The time scales of diffusion and structural rearrangement may be substantially longer than assumed.

5.3. Directions for future research

Kinetic experiments at a range of P and T , to establish the dependence of the decomposition rate on pressure and temperature.

In situ TEM under pressure, to observe precipitate dissociation dynamics in real time.

Calculation of the effect of pressure on the diffusion activation energy (baric effect).

Molecular-dynamics or discrete-dislocation-dynamics modeling to assess the dissociation mechanism within the layered structure.

Refinement of the precipitate structural model via statistical analysis of multiple samples and of electron-microscopy image contrast.

6. Conclusion

In this work, a model is proposed and partially supported in which external hydrostatic loading, by compensating the internal elastic stresses of the outer layer of the silicide precipitate (NiSi_2), can initiate its dissociation. Structural analysis of the inclusions reveals a layered morphology with a varying Si content,

and the kinetic analysis highlights the role of vacancies and diffusion in the decomposition process.

These results contribute to a deeper understanding of the stability mechanisms of metallic and silicide inclusions in silicon crystals and may be useful for the development of doping and contact-formation technologies in microelectronics.

References

- [1] V.I. Fistul, A.R. Turayev, and S.Z. Zainabidinov, *Phys. Status Solidi A* (1993). <https://doi.org/10.1002/pssa.2211360207>
- [2] T.E. Adegoke et al., *ACS Nano* (2024). <https://doi.org/10.1021/acsnano.4c01060>
- [3] P.M. Anderson, J.P. Hirth, and J. Lothe, *Theory of Dislocations*, 3rd ed. (Cambridge University Press, Cambridge, 2017).
- [4] C. Hayzelden and J.L. Batstone, *J. Appl. Phys.* 73, 8279 (1993). <https://doi.org/10.1063/1.353446>
- [5] S.Y. Yoon, S.J. Park, K.H. Kim, and J. Jang, *Thin Solid Films* 383, 34 (2001). [https://doi.org/10.1016/S0040-6090\(00\)01790-9](https://doi.org/10.1016/S0040-6090(00)01790-9)
- [6] A.Q. Soatov and A.R. Turayev, *Sci. Rep. Bukhara State Univ.* 2025/9 (126).
- [7] A.R. Turayev, K.T. Normuratov, and I.M. Soliev, *Sci. Bull. Phys. Math. Res.* 5, No 2, 5 (2023).
- [8] T.Z. Nasirov, A.R. Turayev, M.A. Muydinova, and G.J. Mamatova, *E3S Web Conf.* 458, 01008 (2023). <https://doi.org/10.1051/e3sconf/202345801008>
- [9] Sh.B. Utamurodova, Sh.Kh. Daliev, A.R. Turayev, and F.A. Saparov, in: *Proc. IX Int. Sci. Conf. "Actual Problems of Solid-State Physics,"* Minsk, Belarus, 2021 (Scientific-Practical Materials Research Centre of NAS of Belarus, Minsk, 2021).
- [10] S. Zainabidinov, A.R. Turayev, K.N. Musayev, and N.A. Turgunov, *Inorg. Mater.* 48, No 11, 1195 (2012).
- [11] S. Zainabidinov, A.R. Turayev, N. Turgunov, and A. Kholbekov, *Uzb. J. Phys.* 1, No 4, 334 (1999).
- [12] A.R. Turayev, G. Rakhmonov, A. Soatov, R. Burkhanov, O. Rayimjonov, and N. Botirova, *Tex. J. Multidiscip. Stud.*, 40 (2025).

Investigation of Internal Mechanical Stresses and Piezoresistive Properties of Bi-Sb-Te Based Thin Films Prepared by Thermal Evaporation

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Abstract: This study investigates the formation of internal mechanical stresses in thin semiconductor films and their influence on electrophysical and piezoresistive properties. Thin films based on the (Bi-Sb-Te) solid solution were prepared by vacuum thermal evaporation. Special attention was given to the effects of substrate temperature and deposition conditions on film adhesion, electrical resistivity, and strain sensitivity. A fractional two-stage evaporation process was used to reduce volatile impurities and improve stoichiometric stability. Internal stresses were evaluated by the substrate bending method using the Stoney relation. The results show that the stress state of the films depends strongly on the substrate type and deposition conditions, with both tensile and compressive stress regimes being observed. Optimized (Bi-Sb-Te) thin films exhibited a high gauge factor ($K \approx 42-49$), good linearity of the $(\Delta R/R_0)$ versus strain relationship, and stable temperature behavior. These findings indicate that the developed films are promising for high-precision strain gauge and sensor applications.

1. Introduction.

Thin-film semiconductors play an important role in modern sensor technology, microelectronics, and electromechanical transducers. Compared with bulk materials, thin films usually contain a higher density of structural imperfections such as vacancies, dislocations, grain boundaries, and interfacial defects. These imperfections generate internal mechanical stresses that can significantly modify the electrical and mechanical behavior of the material.

In semiconductor and thermoelectric systems, internal stresses may reach values on the order of 10^8-10^9 Pa. Such stresses influence charge carrier concentration, mobility, defect formation, and the long-term stability of sensor characteristics. Therefore, the study of stress formation mechanisms in thin films is of both fundamental and practical interest.

Among functional semiconductor materials, Bi-Sb-Te-based compounds are widely known for their thermoelectric performance and promising piezoresistive response. However, the properties of thin films derived from these materials depend strongly on the preparation technology, particularly on substrate temperature, evaporation режим, and film-substrate interaction. In this work, the internal mechanical stresses and piezoresistive properties of (Bi-Sb-Te) thin films deposited by thermal evaporation are systematically studied.

2. Theoretical Background and Measurement Principles

Internal stress in thin films can generally be represented as the sum of thermal stress and intrinsic (structural) stress:

$$\sigma = \sigma_{th} + \sigma_{in}$$

2.1. Thermal Stress

Thermal stress arises because of the mismatch between the coefficients of thermal expansion of the film and the substrate. During cooling from deposition temperature to room temperature, this mismatch produces a stress field inside the film. The thermal stress can be expressed as:

$$\sigma_{th} = \frac{E_f}{1 - \mu_f} \int_{T_0}^{T_s} (\alpha_s - \alpha_f) dT$$

where: E_f - is the Young's modulus of the film, μ_f - is the Poisson ratio of the film, α_s - and α_f - are the thermal expansion coefficients of the substrate and the film, respectively, T_s - is the deposition temperature, T_0 - is the reference temperature.

2.2. Stress Determination by the Substrate Curvature Method

The internal stress in the deposited film was experimentally estimated from the curvature of the substrate using the Stoney equation:

$$\sigma = \frac{E_s t_s^2}{6(1 - \nu_s) t_f R}$$

where: E_s - is the Young's modulus of the substrate, ν_s - is the Poisson ratio of the

substrate, t_s - is the substrate thickness, t_f - is the film thickness, R - is the radius of substrate curvature after deposition.

2.3. Piezoresistive Sensitivity

The piezoresistive response of the films was evaluated from the relative resistance change under applied strain:

$$\frac{\Delta R}{R} = K\varepsilon$$

where: ΔR - is the resistance change, R_0 - is the initial resistance, ε - is the applied strain, K - is the gauge factor.

The gauge factor is defined as:

$$K = \frac{\Delta R/R}{\varepsilon}$$

A high value of (K) indicates strong strain sensitivity, which is important for sensor applications.

3. Experimental Procedure

Thin films based on the (Bi-Sb-Te) solid solution were deposited by vacuum thermal evaporation. To improve film purity and compositional stability, a fractional evaporation method was used. In this procedure, the first 10% of the evaporated flux, which may contain more volatile impurities, was excluded from deposition. The next 80% of the evaporated material was used for film formation, while the

final 10% remained in the crucible. This approach reduced the influence of nonstoichiometric evaporation and improved reproducibility.

The films were deposited onto different substrates, including glass-ceramic (CNO) and polyimide tape. The substrate temperature was varied to determine its influence on microstructure, adhesion, electrical properties, and internal stress.

Electrical resistivity and strain sensitivity were measured after deposition. The piezoresistive characteristics were obtained by recording the relative resistance change as a function of mechanical deformation. The linearity, reversibility, and repeatability of the response were analyzed using loading-unloading cycles.

4. Results and Discussion

4.1. Compositional Analysis

Microstructural and compositional investigations showed that the deposited thin films preserved the elemental ratio of the bulk source material with only minor deviations. This confirms the effectiveness of the fractional evaporation technique in maintaining stoichiometric stability.

Element	Bulk Material (%)	Thin Film (%)
Bismuth (Bi)	16.12	16.06
Antimony (Sb)	26.90	26.32
Tellurium (Te)	57.00	56.81

The slight reduction in Sb and Te content in the film may be related to their partial re-evaporation or differences in vapor pressure during the deposition process. Nevertheless, the deviations remain small, indicating that the film composition closely matches the starting material.

4.2. Influence of Substrate Temperature on Electrical Resistivity

One of the most important technological dependences observed in this study was the variation of electrical resistivity with substrate temperature. At relatively low substrate temperatures ($T_s < 150^\circ\text{C}$), the films exhibited high resistivity due to insufficient crystallization, a high density of structural

defects, and enhanced carrier scattering at grain boundaries.

As the substrate temperature increased, the film quality improved significantly. At approximately ($T_s \approx 200^\circ\text{C}$), the resistivity reached an optimal low value, close to ($2.2 \cdot 10^{-6} \Omega \cdot \text{m}$). This indicates improved crystallinity, better intergrain connectivity, and stronger adhesion to the substrate. Thus, the substrate temperature around (200°C) can be considered optimal for obtaining films with favorable electrophysical properties.

4.3. Internal Mechanical Stresses

The stress state of the deposited films depended strongly on both the substrate material and the deposition conditions. Calculations based on the substrate curvature

method revealed that the films could exhibit either tensile or compressive stress. This transition is mainly associated with the mismatch in thermal expansion coefficients and differences in the elastic properties of the substrate-film system.

In addition to thermal effects, intrinsic stresses caused by the growth mechanism, grain coalescence, point defects, and structural relaxation also contributed to the overall stress state. Since internal stress influences charge carrier transport and defect structure, its control is essential for achieving stable and reproducible piezoresistive characteristics.

4.4. Piezoresistive Characteristics

The Bi-Sb-Te films demonstrated a clear piezoresistive response under applied mechanical strain. The dependence of $\Delta R/R_0$ on strain ε was found to be approximately linear over the investigated range, which is a desirable feature for strain gauge applications. Furthermore, the loading and unloading curves showed good reversibility, indicating stable film behavior under repeated deformation.

The measured gauge factor reached values in the range ($K \approx 42\text{--}49$), which is considerably high for practical strain-sensitive materials. Experimental results also suggested that the sensitivity under tensile strain was greater than under compressive strain. This asymmetry can be explained by the presence of pre-existing internal stresses and the polycrystalline structure of the films, which affect the redistribution of local deformation and carrier transport pathways.

The linear relationship, combined with acceptable measurement uncertainty and

reproducible behavior, confirms that these films have significant potential for precision sensing applications.

5. Conclusion

The present study demonstrates that the internal mechanical stresses and piezoresistive properties of Bi-Sb-Te-based thin films are strongly determined by deposition technology and substrate conditions.

The main conclusions are as follows:

Internal stresses in the films originate from both thermal mismatch and intrinsic structural factors, and their magnitude and sign depend on the substrate material and deposition regime.

The use of a fractional two-stage evaporation process is effective in suppressing volatile impurities and preserving the stoichiometric composition of the deposited films.

The substrate temperature has a decisive influence on film quality. Films deposited at approximately (200 °C) exhibit improved crystallinity, lower resistivity, and better adhesion.

The investigated films show a strong and nearly linear piezoresistive response, with a high gauge factor in the range of (42–49), making them suitable for strain gauge and sensor applications.

Due to their high sensitivity and stability, (Bi-Sb-Te) thin films are promising candidates for use in advanced microelectromechanical systems (MEMS) and high-precision deformation sensors.

References

- [1]. Stoney, G. G. *The tension of metallic films deposited by electrolysis*. Proceedings of the Royal Society A, 1909, 82(553). DOI: 10.1098/rspa.1909.0021. ([Royal Society Publishing](#))
- [2]. Feng, X., Huang, Y., Jiang, H., Ngo, D., Rosakis, A. J. *The effect of thin film/substrate radii on the Stoney formula for thin film/substrate subjected to nonuniform axisymmetric misfit strain and temperature*. Journal of Mechanics of Materials and Structures, 2006. ([Jiang Hanqing Lab](#))
- [3]. Janssen, G. C. A. M., Abdalla, M. M., van Keulen, F., Pujada, B. R., van Venrooy, B. *Celebrating the 100th anniversary of the Stoney equation for film stress: Developments from*

polycrystalline steel strips to single crystal silicon wafers. Thin Solid Films, 2009. ([ScienceDirect](#))

- [4]. Winkler, M., Liu, X., König, J. D., Buller, S., Schürmann, U., Kienle, L., Bensch, W., Böttner, H. *Electrical and structural properties of Bi₂Te₃ and Sb₂Te₃ thin films grown by the nanoalloying method with different deposition patterns and compositions*. Journal of Materials Chemistry, 2012, 22, 11323. DOI: 10.1039/C2JM30363A. ([RSC Publishing](#))
- [5]. Tang, X., Li, Z., Uher, C., et al. *A comprehensive review on Bi₂Te₃-based thin films: Thermoelectrics and beyond*. Interdisciplinary Materials, 2022. DOI: 10.1002/idm2.12009. ([Wiley Online Library](#))
- [6]. Cao, T., He, W., Chen, Y., et al. *Advances in bismuth-telluride-based thermoelectric devices: Progress and challenges*. Carbon Energy, 2023. ([ScienceDirect](#))

- [7].Liu, Y., et al. *Anomalous thermoelectricity in strained Bi₂Te₃ films*. Scientific Reports, 2016, 6, 32661. ([Nature](#))
- [8].Azimov T., Gajazarova K., Onarkulov K. Method for determining the contact resistance of thermoelements //Euroasian Journal of Semiconductors Science and Engineering. – 2020. – T. 2. – №. 5. – C. 11.
- [9].Onarkulov K. E., Azimov T. M. Study of diffusion processes in contact areas of thermocouples with metals //International Journal of Physical Sciences. – 2023. – T. 18. – №. 1. – C. 59-64.
- [10].Yusupova D., Azimov T. Electrophysical and thermoelectric properties of thin-film materials based on bismuth, tellurium, antimony and selenium compounds //AIP Conference Proceedings. – AIP Publishing LLC, 2026. – T. 3401. – №. 1. – C. 020022.
- [11].Onarkulov K., Azimov T. Diffusion processes in contact areas thermoelements based on Pb-Sb alloys //AIP Conference Proceedings. – AIP Publishing LLC, 2024. – T. 3045. – №. 1. – C. 030096.

Особенности электрических свойств шлакощелочных композиций наполненных техническим углеродом в рамках теории протекания

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I. Введение.

В последние годы отмечается возобновление научного интереса к композиционным электропроводящим материалам на основе цементных матриц, модифицированных электропроводящими наполнителями. В ряде недавних исследований [1, 2] установлено, что подобные системы обладают новыми функциональными свойствами, в том числе способностью к эффективному поглощению электромагнитного излучения. Вместе с тем закономерности формирования их электрофизических характеристик в зависимости от структуры и состава композиции остаются недостаточно изученными. В этой связи применение современных представлений теории гетерогенных систем для анализа указанных зависимостей представляется актуальным и научно обоснованным.

Настоящая работа посвящена разработке и исследованию электрофизических свойств композиционных материалов на основе шлакощелочного вяжущего наполненного техническим углеродом и выяснению их модельной структуры в рамках теории гетерогенных систем.

II. Методология исследования.

В качестве объекта исследования использовали электростале-плавильный шлак, растворимое жидкое стекло и технический углерод размером частиц 10 мкм. Объемную долю наполнителя варьировали от 0,02 до 0,6. Композиции готовили перемешиванием частиц

вяжущего и технического углерода в шаровой агатовой мельнице в течении 7 часов. Изготовлением тестообразной массы из композиции путем добавления 3-5% растворимого жидкого стекла в пересчете на Na_2O от массы шлака формованием образцов диаметром 15 мм и высотой 2 мм при давлении 3 МПа. Образцы подвергались тепло влажностной обработке при $t = 90 \pm 5^\circ C$, и режиме 3 + 6 + 3 ч.

III. Анализ и результаты.

Результаты измерения электропроводности σ композитов в зависимости от объемных долей наполнителя V_1 представлены точками на рис.1. В области $V_1 = 0,15 \div 0,3$ наблюдается резкий рост электропроводности от значения σ_2 матрицы до значения σ_1 электропроводности наполнителя. Критическая концентрация V_c (порог протекания), при которой впервые образуется бесконечный кластер (БК) из частиц наполнителя определялась дифференцированием $lg\sigma$ композита по V_1 (рис.1, а), значения которой равна $V_c = 0,2$.

а) Зависимость $\frac{d lg \sigma}{d V_1}$ от объемной доли наполнителя V_1 .

Известно [3], что электропроводность двухфазных систем, состоящих из компонентов с резко отличающимися проводимостями описываются степенными формулами:

$$\sigma(V_1) = \sigma_2 \left(\frac{V_c - V_1}{V_c} \right)^{-q} \quad \text{при} \quad V_1 < V_c \quad (1)$$

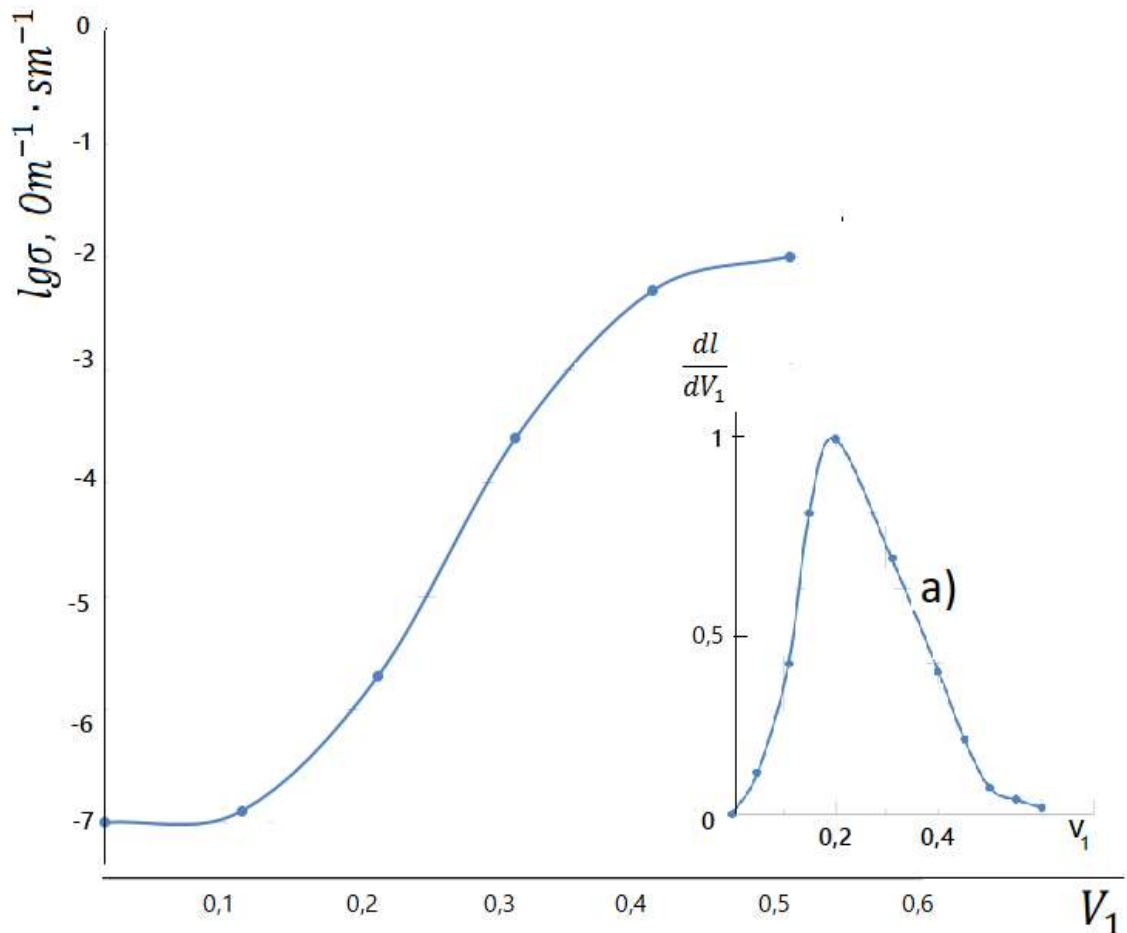


Рис.1. Зависимость электропроводности композитов на основе шлакощелочного вяжущего объемной от доли V_1 наполнителя (технический углерод).

$$\sigma(V_1) = \sigma_1 \left(\frac{V_1 - V_c}{1 - V_c} \right)^t \quad \text{при} \quad V_1 > V_c \quad (2)$$

$$\sigma(V_1) = h^s \sigma_1 \quad \text{при} \quad V_1 = V_c \quad (3)$$

где σ_1 – проводимость наполнителя, σ_2 – проводимость связующего, $h = \frac{\sigma_2}{\sigma_1}$, показатели степени t , q и s в (1), (2), (3) как и в теории фазовых переходов названы критическими индексами, значение которых для трехмерных систем, соответственно равны 1,7; 0,98; 0,62. Между критическими индексами существует соотношение

$$q = t \left(\frac{1}{s} - 1 \right) \quad (4)$$

аналогичное соотношение теории подобия и фазовых переходов.

Для определения параметров формул (1) и (2) экспериментальные

результаты строились в координатах $\lg \sigma - \lg \left(\frac{V_1 - V_c}{1 - V_c} \right)$ и $\lg \sigma - \lg \left(\frac{V_c - V_1}{V_c} \right)$ (рис.2). По тангенсу угла наклона из графика зависимости $\lg \sigma - \lg \left(\frac{V_c - V_1}{V_c} \right)$ определен критический индекс q , значение которого для композиции равен 1,16. Как видно из рис.2, для композиции на основе шлакощелочного вяжущего и технического углерода в интервале V_1 зависимость $\lg \sigma - \lg \left(\frac{V_1 - V_c}{1 - V_c} \right)$ имеет линейный характер. Из наклона зависимости определено значение $t=2,3$. Эти величины несколько отличаются от приведенных в [3, 4], но вместе с тем, как и [2, 3] так и в нашей работе, соотношение (4) выполняется с точностью, не выходящей за пределы точности расчета самых индексов (s равен 0,67).

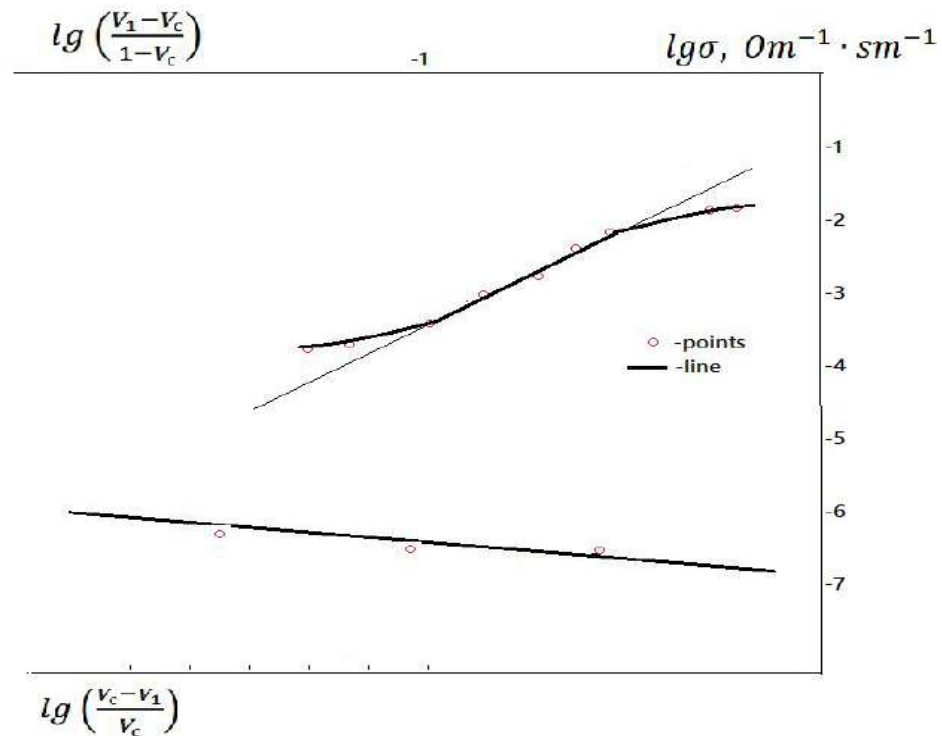


Рис.2. Зависимость электропроводности композиции на основе шлакощелочного вяжущего наполненного техническим углеродом от объемной доли наполнителя V_1 в координатах: $\lg \sigma$ от $\lg \left(\frac{V_1 - V_c}{1 - V_c} \right)$; $\lg \sigma$ от $\lg \left(\frac{V_c - V_1}{V_c} \right)$.

Экстраполируя прямолинейный участок зависимости $\lg \sigma$ — $\lg \left(\frac{V_1 - V_c}{1 - V_c} \right)$ рис.2. при $V_1 \rightarrow 1$ определена электропроводность БК в композиции, значение которой оказалось равным $\sigma_1 = 0,06 \text{ Ом}^{-1} \cdot \text{см}^{-1}$. А значение электропроводности прессованного порошка технического углерода полученного при давлении 3 МПа имело $\sigma = 1 \text{ Ом}^{-1} \cdot \text{см}^{-1}$. Полученный результат σ_1 оказался более низким по сравнению с соответствующими значениями σ прессованных порошков технического углерода. Это можно объяснить контактным сопротивлением между частицами наполнителя.

Таким образом, в композиционных материалах σ в основном определяется свойствами изолирующих пленок между контактирующими частицами наполнителя. В таких системах в отличие от модельных систем, рассматриваемых в задачах теории протекания, где контактное сопротивление

между соприкасающимися элементами проводящих частиц не учитывается (удельное протекания), наблюдается так называемое «контактное протекание».

В этом случае вблизи порога протекания $\sigma(V_1)$ двухкомпонентных систем также, как было показано выше, описывается формулами (1), (2) и (3). Близость экспериментальных значений критических индексов с расчетными показывает, что представления о топологии бесконечного кластера [4] могут быть использованы и в случае контактного протекания.

В таблице 1 приведены значения $P(V_1)$ вычисленные по формуле

$$P(V_1) = D \left(\frac{V_1 - V_c}{1 - V_c} \right)^\beta$$

где D — численный коэффициент порядка единицы, β — критический индекс, для трехмерных систем 0,4 объемная доля БК вычисленная по формуле [4]

$$V_1' = \frac{V_1 - V_c}{1 - V_c}$$

в зависимости от V_1 . Значение $P(V_1)$ при удалении от порога протекания в сторону больших V_1 возрастает. Это означает, что бесконечный кластер постепенно

присоединя конечные кластеры, образованные между частицами технического углерода становиться более «густым». Величина V_1 как видно из таблицы, вблизи порога протекания составляет ничтожную часть V_1 .

Таблица 1. Зависимость параметров, характеризующих структуру бесконечного кластера от объемной доли наполнителя V_1 для композиций на основе шлакощелочного вяжущего наполненного техническим углеродом.

V_1	V_1'	$P(V_1)$	Z/R	l'	s_1'	V_1''	V_1'''
0,22	0,025	0,23	6,3	1,26	$1,05 \cdot 10^{-3}$	$3,8 \cdot 10^{-3}$	$2,12 \cdot 10^{-2}$
0,27	0,068	0,34	3,8	0,76	$9,7 \cdot 10^{-3}$	$6,53 \cdot 10^{-2}$	$2,7 \cdot 10^{-2}$
0,3	0,13	0,44	2,7	0,554	$1,7 \cdot 10^{-2}$	$9,97 \cdot 10^{-2}$	$3,03 \cdot 10^{-2}$
0,35	0,19	0,51	2,29	0,46	$1,06 \cdot 10^{-1}$	$1,59 \cdot 10^{-1}$	$3,15 \cdot 10^{-2}$
0,37	0,21	0,54	2,18	0,44	$1,4 \cdot 10^{-1}$	$1,75 \cdot 10^{-1}$	$3,5 \cdot 10^{-2}$
0,4	0,29	0,61	1,85	0,37	$2,33 \cdot 10^{-1}$	$2,49 \cdot 10^{-1}$	$4,1 \cdot 10^{-2}$
0,47	0,34	0,65	1,71	0,34	$3,4 \cdot 10^{-1}$	$2,98 \cdot 10^{-1}$	$4,16 \cdot 10^{-2}$
0,5	0,38	0,68	1,62	0,32	$7,4 \cdot 10^{-1}$	$3,36 \cdot 10^{-1}$	$4,4 \cdot 10^{-2}$

Значения Z/R вычисленные по формуле [4]

$$\frac{Z}{R} = \left(\frac{V_1 - V_c}{1 - V_c} \right)^{-(\xi - \nu)},$$

где $\xi = t - \nu$, $t = 2,3$ и $\nu = 0,9$ показывает, что вблизи порога протекания бесконечный кластер очень извилист. Зная длину скелета L и σ_1 сопротивление композиций, по формуле $s' = \frac{1}{L} \frac{l'}{\sigma_1}$ вычислили среднее сечение БК, где $l' = \frac{Z}{R} l$ ($l = 2 \text{ mm}$ – толщина образца) и объемную долю скелета V_1'' в объеме sl , где s – сечение образца. Значение объемной доли скелета V_1'' в единичном объеме и доля мертвых концов $V_1''' = V_1' - V_1''$ для образцов приведены в таблице. Вблизи порога протекания объемная доля скелета,

принадлежавшего к бесконечному кластеру, составляет всего 1-5% по сравнению с долей мертвых концов. Аналогичные результаты были получены для полимерных композиций [5].

IV. Заключение.

Анализ и обработка полученных экспериментальных данных показывают, что электрофизические свойства композиционных материалов на основе шлакощелочных вяжущих, модифицированных техническим углеродом, в значительной степени определяются контактными явлениями, возникающими между электропроводящими частицами наполнителя.

ЛИТЕРАТУРА

[1]. Chao Ma, Shuai Xie, Zihao Wu, Zhijiang Ji, Jing Wang Research and simulation of electromagnetic wave absorbing performance for lightweight cement-based materials containing expanded polystyrene with different particle sizes

and carbon black.// <https://doi.org/10.1016/j.conbuildmat.2024.136496>

[2]. Конев О.Б. Быстротвердеющие шлакощелочные вяжущие и бетоны на основе кристаллических шлаков для изделий, формируемых полусухим прессованием. //Автореферат к.т.н. – Макеевка. 2020.26 с.

- [3]. M. Karabayeva, B.M. Matyakubov, D. Saidkulov, Y.M. Raximova, Sh. Kamilov and U. Abduraxmanov Transport of Charge Carriers in Organic Disordered Semiconductors Based on Polyacrylonitrile / Phys. Chem. Res., Vol. 13, No. 1, 31-36, March 2025.
- [4]. Д. Саиткулов, М. Карабаева, Я. Рахимова, Ш. Камилов, У. Абдурахмонов. Перенос носителей заряда в гетерогенных материалах содержащих электропроводящие наночастицы. O'zMU xabarlari NUUz FIZIKA 3/2 2024, стр. 556-558.
- [5]. Д.Саиткулов, У. Абдурахманов, Я. Рахимова, Ш. Камилов Электропроводность композиционных полимерных материалов на основе полиэтилена содержащих наночастицы меди. O'zMU xabarlari ACTA NUUz FIZIKA 3/2 2025., стр. 483-486

Босимли сув қувурлари учун фазовий буралган эллиптик куракли микрогидротурбина роторини кўп мезонли оптималлаштириш

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Аннотация: Босимли сув узатиш, ирригация ва саноат қувур тармоқларида талабдан ортиқча босим энергияси босимни камайтирувчи арматура ҳамда маҳаллий гидравлик қаршиликларда кўпинча қайтариб бўлмас гидравлик йўқотиш сифатида сўнади. Мазкур тадқиқотда қувурнинг асосий сув ўтказиш қобилиятини сақлаган ҳолда ушбу энергияни рекуперация қилиш учун маркази очик ва фазовий буралган эллиптик куракли микрогидротурбина ротори таклиф этилди. Тадқиқот аналитик геометрик-гидродинамик моделлаштириш, кўп мезонли параметрик оптималлаштириш ва COMSOL Multiphysics 6.3 муҳитидаги уч ўлчамли CFD моделлаштиришни ягона ҳисоблаш занжирига бирлаштиради. Аналитик оптималлаштириш натижасида эллиптик ташқи габарит ярим ўқларининг нисбати $a/R_p = 1$, $b/R_p = 0.68804$ ва $a/b = 1.45340$ бўлганда, роторнинг келтирилган момент потенциали сферик базавий кўринишга нисбатан 12.04% га ортиши аниқланди. Баландлик бўйича айлантирувчи момент юкланишининг нотекислиги ҳисобга олинганда, пульсациянинг тўртинчи гармоникаси учун вазнланган аналитик ноль қиймати $\Phi_{opt} \approx 182.78^\circ$ бурчада ҳосил бўлди ҳамда $\Phi = 180^\circ$ амалий сонли моделлаштириш геометрияси сифатида қабул қилинди. Номинал иш режимида $V = 2.0 \text{ m/s}$ ва $\Delta p = 0.20 \text{ MPa}$ режимда аналитик энергетик модель $P_{shaft} = 2390.8 \text{ W}$, $\bar{M} = 54.52 \text{ N m}$ ва $\eta_h = 0.761$ кўрсаткичларини таъминлади. Сонли моделлаштириш SST $k - \omega$ турбулентлик модели, ҳаракатланувчи тўр ва квази-стационар «музлатилган ротор» (Frozen Rotor) ёндашувлари асосида $r_{c0} = 21 - 33 \text{ mm}$ параметрик ўзгариш диапазонида бажарилди.

Калит сўзлар: қувур ичи гидротурбинаси, эллиптик курак, микрогидроэнергетика, босим энергиясини рекуперация қилиш, кўп мезонли оптималлаштириш, SST $k - \omega$, сонли гидродинамика (CFD), айлантирувчи момент пульсацияси.

Аннотация: В напорных сетях водоснабжения, ирригации и промышленных трубопроводах избыточная энергия давления часто необратимо рассеивается на редукционной арматуре и местных гидравлических сопротивлениях. В данном исследовании предложен ротор микрогидротурбины с пространственно закрученными эллиптическими лопастями и центральной открытой областью для рекуперации этой энергии при сохранении основной водопропускной способности трубы. Работа объединяет аналитическое геометрико-гидродинамическое моделирование, многокритериальную параметрическую оптимизацию и трехмерное CFD-моделирование в среде COMSOL Multiphysics 6.3 в единую расчетную цепочку. Аналитическая оптимизация показала, что при соотношении полуосей эллиптической образующей $a/R_p = 1$, $b/R_p = 0.68804$ и $a/b = 1.45340$ приведенный моментный потенциал ротора увеличивается на 12.04% по сравнению со сферической базовой конфигурацией. С учетом неравномерного распределения крутящего момента по высоте ротора взвешенный аналитический ноль четвертой гармоники пульсаций достигнут при $\Phi_{opt} \approx 182.78^\circ$, а угол закрутки $\Phi = 180^\circ$ принят в качестве практической геометрии для численного моделирования. Для номинального режима $V = 2.0 \text{ m/s}$ и $\Delta p = 0.20 \text{ MPa}$ аналитическая энергетическая модель обеспечила $P_{shaft} = 2390.8 \text{ W}$, $\bar{M} = 54.52 \text{ N m}$ и $\eta_h = 0.761$. Численное моделирование выполнено с применением модели турбулентности SST $k - \omega$, подвижных сеток и квазистационарного подхода «замороженного ротора» (Frozen Rotor) в диапазоне параметрического варьирования $r_{c0} = 21 - 33 \text{ mm}$.

Ключевые слова: внутритрубная гидротурбина, эллиптическая лопасть, микрогидроэнергетика, рекуперация энергии давления, многокритериальная оптимизация, SST $k - \omega$, гидродинамическое моделирование (CFD), пульсация крутящего момента.

Abstract: In pressurized water supply, irrigation, and industrial pipeline networks, excess pressure energy is frequently dissipated irreversibly across pressure-reducing valves and local hydraulic resistances. In this study, a micro-hydropower turbine rotor featuring a central open core and spatially twisted elliptical blades is proposed to recover this energy while preserving the primary water conveyance capacity of the pipe. The research integrates analytical geometric-hydrodynamic modeling, multi-objective parametric optimization,

and three-dimensional CFD modeling in COMSOL Multiphysics 6.3 into a unified computational framework. Analytical optimization demonstrated that semi-axis ratios of the elliptical envelope of $a/R_p = 1$, $b/R_p = 0.68804$ and $a/b = 1.45340$ enhance the reduced torque potential of the rotor by 12.04% compared to the spherical baseline configuration. Accounting for the non-uniform vertical distribution of the torque load, the weighted analytical zero of the fourth pulsation harmonic was obtained at an angle of $\Phi_{opt} \approx 182.78^\circ$, and $\Phi = 180^\circ$ was adopted as the practical geometry for numerical simulation. Under nominal operating conditions ($V = 2.0$ m/s, $\Delta p = 0.20$ MPa), the analytical energy model yielded $P_{shaft} = 2390.8$ W, $\bar{M} = 54.52$ N m and $\eta_h = 0.761$. Numerical modeling was carried out using the SST $k - \omega$ turbulence model, Moving Mesh, and the quasi-steady Frozen Rotor formulation within a parametric sweep range of $r_{c0} = 21 - 33$ mm.

Keywords: in-pipe hydro turbine, elliptical blade, micro-hydropower, pressure energy recovery, multi-objective optimization, SST $k - \omega$, computational fluid dynamics (CFD), torque ripple.

I. Кириш

Сув таъминоти, ирригация ва саноат қувурларида мавжуд ортиқча гидравлик босим кўп ҳолларда ростловчи арматура ва маҳаллий қаршиликларда энергия йўқотилишига олиб келади. Кичик гидроэнергетик технологиялар, сув тармоқларида энергияни қайта олиш стратегиялари ва амалий объектлар бўйича умумлаштирувчи тадқиқотлар ушбу захирадан фойдаланишнинг энергетик аҳамиятини кўрсатади. Бироқ техник ечимни танлашда қувват олиш билан бир қаторда истеъмолчига зарур босим ва сув сарфини сақлаш ҳам ҳал қилувчи шарт ҳисобланади [1–3].

Ортиқча босим энергиясини ўзлаштиришнинг кенг ўрганилган йўналишларидан бири насосларни турбина режимида ишлатишдир. Мавжуд тадқиқотларда бундай агрегатларни танлаш, иш режимларини ростлаш, ўлчамларини мақбуллаштириш ва иқтисодий самарадорлигини баҳолаш усуллари ишлаб чиқилган. Уларнинг умумий натижаси шуки, энергетик фойда агрегатнинг фақат номинал самарадорлигига эмас, балки ўзгарувчан сарф ва босимдаги мослашувчанлигига ҳам боғлиқ; айти пайтда конструктив ўлчам ва қувурдаги гидравлик қаршилик ихчам қурилмалар учун чеклов бўлиб қолади [4–6].

Қувур ичида бевосита жойлаштириладиган сферик, Дарье ва Савониус турбиналарига оид назарий ва тажрибавий ишлар ротор шакли, оқим билан ўзаро таъсир ва айлантирувчи момент ҳосил бўлишининг ўзига хослигини ёритади. Ушбу ечимларда энергия ажратиш олиш

имконияти мавжуд бўлса-да, чекланган қувур кесимида ротор туфайли оқимнинг қайта тақсимланиши ва унинг ортидаги гирдобли соҳаларни ҳисобга олиш зарурати сақланади [7–9].

Гидрокинетик турбиналарга бағишланган тадқиқотлар ротор билан қувур девори орасидаги тирқиш ҳамда оқим кесимининг тўсилиш даражаси қувват олиш ва қўшимча босим йўқотиши ўртасидаги мувозанатни белгилашини кўрсатади. Тўсилиш таъсирини ҳисоблаш ва тузатиш усуллари натижаларни қиёслаш учун муҳим бўлса-да, улар куракнинг фазовий геометриясини мустақил равишда мақбуллаштириш масаласини тўлиқ ҳал этмайди [10–12]. Курак тури ва эллиптик шаклли кўп босқичли роторлар бўйича изланишлар эса курак профилининг энергия олиш хусусиятлари ва оқим тузилишига таъсирини асослайди [13–14].

Сув узатиш қувурларида энергияни қайта олишга қаратилган амалий ечимлар, босимни турбина орқали ростлаш тизимлари ва тажрибадан ўтказилган қувур ичи қурилмалари техник татбиқ имкониятларини намоён этган. Шу билан бирга, турли сарфларда барқарор ишлаш, ўрнатиш қулайлиги ва мавжуд тармоқнинг сув ўтказиш қобилиятига таъсирни чеклаш масалалари конструкцияни мақсадли такомиллаштириш заруратини кўрсатади [15–17].

Тармоқ миқёсидаги тадқиқотларда энергияни қайта олиш ва сув йўқотишларини биргаликда камайтириш, агрегатни мақбул танлаш ҳамда ўзгарувчан режимлар учун бошқарув алгоритмларини ишлаб чиқишга эътибор қаратилган. Бу

ишлар тизим даражасидаги оптималлаштиришнинг афзаллигини кўрсатса-да, роторнинг алоҳида кураклари геометрияси, босим пасайиши ва айлантирувчи момент тебранишини боғловчи ягона конструктив мезонларни етарлича очиб бермайди [18–21]. Насосларни турбина режимида қўллашнинг умумлаштирилган таҳлили ва қувур ичи параллел ўқли роторларининг дастлабки синовлари ҳам ихчам муқобил конструкцияларни ривожлантириш долзарблигини тасдиқлайди [22–23].

Шундай қилиб, мавжуд изланишлар босим энергиясини қайта олишнинг умумий самарадорлигини ва айрим турбина шакллариининг гидродинамик хусусиятларини асослаган бўлса-да, маркази очик, фазовий буралган эллиптик куракли қувур ичи роторларида эллиптик ярим ўқлар нисбати, куракнинг радиал жойлашуви, ватарнинг баландлик бўйича тақсимооти, буралиш бурчаги ва кураклар сонини ўзаро боғлиқ ҳолда танлаш масаласи етарли даражада тизимлаштирилмаган. Айниқса, айлантирувчи момент, унинг даврий тебраниши ва қўшимча босим йўқотишини ягона кўп мезонли ҳисоблаш доирасида мувофиқлаштириш илмий-амалий аҳамиятга эга.

Мазкур тадқиқотнинг мақсади — босимли сув қувурлари учун маркази очик, фазовий буралган эллиптик куракли микрогидротурбина роторининг геометрик параметрларини аналитик гидродинамик модел, кўп мезонли оптималлаштириш ва уч ўлчамли сонли оқим таҳлилини уйғунлаштириш асосида танлаш ҳамда унинг энергетик кўрсаткичлари ва момент тебранишига таъсирини баҳолашдан иборат.

II. Материаллар ва методология

Мазкур тадқиқотда дастлаб эллиптик ротор геометрияси, курак ватари ва унинг радиал жойлашувини боғловчи аналитик геометрик-гидродинамик модел ишлаб чиқилди. Сўнгра роторнинг айлантирувчи момент потенциалини максималлаштириш ҳамда тўртинчи гармоника пульсациясини минималлаштириш шартлари бўйича кўп мезонли оптималлаштириш амалга оширилди. Якуний босқичда эса COMSOL

Multiphysics муҳитида уч ўлчамли сонли гидродинамик (CFD) моделлаштириш воситасида олинган аналитик оптимал ечимлар реал гидродинамик ва геометрик чекловлар доирасида текширилди.

2.1. Аналитик геометрик-гидродинамик модель

Тадқиқотнинг ҳисоблаш занжири эллиптик габарит ва курак геометриясини аналитик параметрлаш, гидродинамик саралаш, кўп мезонли танлов ҳамда 3D CFD текширувини бирлаштиради. Ички диаметри D_p , радиуси $R_p = D_p/2$ бўлган қувурдаги ўртача оқим тезлиги V ва сарф Q учун кесим юзаси $A_p = \frac{\pi D_p^2}{4}$, Рейнольдс сони $Re = \frac{\rho V D_p}{\mu}$ ҳамда тикланиши мумкин бўлган назарий гидравлик қувват $P_h = \Delta p Q$ кўринишида аниқланади.

Эллиптик ротор меридионал контури a катта ва b кичик ярим ўқлари орқали параметрланди ($H_r = 2a$ ва $D_r = 2b$). Баландлик бўйича ташқи радиус ва кўндаланг кесимни тўсиш даражаси куйидагича ифодаланади:

$$r_e(z) = b \sqrt{1 - \frac{z^2}{a^2}}, \quad -a \leq z \leq a$$

$$\beta_{sw} = \frac{A_{sw}}{A_p} = \frac{4ab}{D_p^2} \quad (1)$$

Курак ватарининг баландлик бўйича ўзгариши четки c_h ва марказий c_m ватарлар орқали қабул қилинди:

$$c(z) = c_h + (c_m - c_h) \left[1 - \left(\frac{z}{a} \right)^2 \right]^{v_c} \quad (2)$$

Базавий CFD модели учун $c_h = 8 \text{ mm}$, $c_m = 20 \text{ mm}$, ва шакл кўрсаткичи $v_c = 1$ қабул қилинди.

Қувурдаги турбулент тезлик профили таъсирини баҳолаш учун нисбий радиал координата $\xi = r_{co}/R_p$ кўринишда киритилди ва куйида келтирилган гидродинамик функция қабул қилинди:

$$\Psi_M(\xi) = \xi(1 - \xi)^{2/m_v} \quad (3)$$

Унинг стационар максимум шартидан

$$\xi_{hyd} = \frac{m_v}{m_v + 2}$$

келиб чиқади; $m_v = 7$ бўлганда ($\xi_{hyd} = 0.7778$ $r_{co,hyd} = 38.89 \text{ mm}$) бўлади.

Тўрт куракли ($Z_b = 4$) роторда баландлик бўйича нотекис момент юкланиши $w_M(u) = (1 - u^2)^{5/2}$, $u = z/a$ ҳисобга олинганда, тўртинчи гармониканинг нормаллаштирилган коэффициенти 3-тартибли биринчи тур Бессель функцияси $J_3(\beta)$ орқали ифодаланади:

$$G_4(\Phi) = \frac{48J_3(\beta)}{\beta^3}, \quad \beta = 2\Phi_{rad} \quad (4)$$

тенглама бўйича вазнланган биринчи аналитик ноль $\Phi_{opt} \approx 182.78^\circ$ да ҳосил бўлиб, амалий CFD таҳлили учун $\Phi = 180^\circ$ танланди. Айланма тезлик коэффициенти куйидагича ҳисобланди:

$$\lambda = \frac{\omega R_{eff}}{V}$$

$$P_{shaft} = \bar{M}\omega, \quad \eta_h = \frac{P_{shaft}}{\Delta p Q}$$

2.2. Кўп мезонли оптималлаштириш стратегияси

Оптималлаштиришда мақсад функциялари ва чекловлар куйидагича шакллантирилди:

$$\max\{P_{shaft}, \eta_h, \bar{M}\}, \quad \min\{\Delta p_{loss}, K_{M,ripple}\} \quad (5)$$

бунда ватарнинг маҳаллий радиалга нисбатан бурчаги $\gamma = -40^\circ$, кураклар сони $Z_b = 4$ таянч қийматлар сифатида қабул қилинди. Номинал иш режимида $V = 2.0$ m/s, $\Delta p = 0.20$ МПа, $R_{eff} = 0.045612$ m аналитик модел $\omega_{opt} = 43.848$ rad/s, $\bar{M} = 54.52$ N·m, $P_{shaft} = 2390.8$ W ва $\eta_h = 0.761$ қийматларини берди. Бу қийматлар аналитик дизайн натижаси бўлиб, уларни COMSOL transient signal ёки экспериментал ўлчов деб талқин қилиш мумкин эмас. Ностационар аналитик прогнозда ideal Fourier фазалаш dominant harmonic suppression ни баҳолайди; реал M_{rms} , ΔM_{pp} , p_{rms} ва wake u_{rms} қийматлари эса Time Dependent Sliding Mesh CFD ёки стенд синовидан олиниши керак.

2.3. Уч ўлчамли CFD модели ва чегаравий шартлар

Сонли гидродинамик тадқиқот COMSOL Multiphysics 6.3 муҳитида амалга оширилди. Модель кўзғалмас қувур ва айланувчи ротор соҳаларига ажратилиб, улар чегаравий интерфейс жуфтлиги билан туташтирилди. Турбулентликни моделлаштириш учун девор яқинидаги ва эркин оқим соҳаларини

юқори аниқликда ҳисобловчи SST $k - \omega$ модели ҳамда квази-стационар Frozen Rotor ёндашуви танланди.

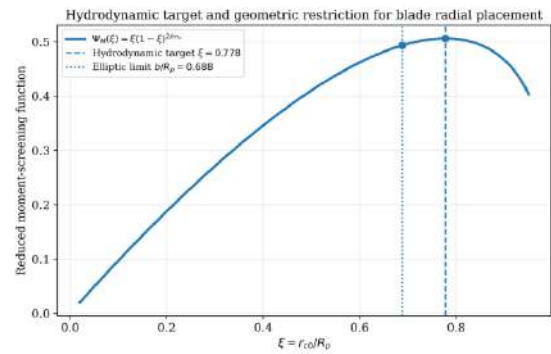
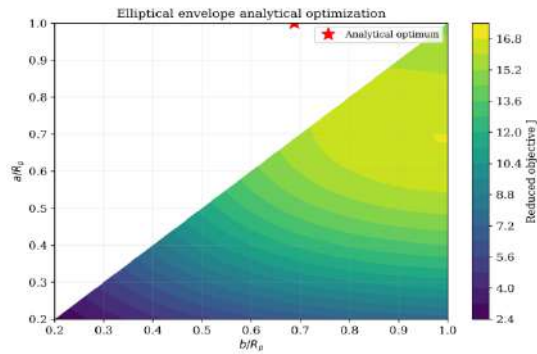
Асосий 3D parametrik model учун $D_p = 100$ mm, $R_p = 50$ mm, $\delta = 2$ mm, $a = 45$ mm, $b = 34.7$ mm, $D_r = 69.4$ mm ва $Z_b = 4$ қабул қилинди. CFD geometry reference конфигурациясида $c_h = 8$ mm, $c_m = 20$ mm, $v_c = 1$, blade thickness $t_b = 1$ mm, $\gamma = -40^\circ$ ва $\omega = 40$ rad/s киритилди. Кириш ва чиқиш абсолют босимлари мос равишда 6.0795×10^5 Pa ва 4.0530×10^5 Pa бўлиб, ишчи босим фарқи 2.0265×10^5 Pa ни ташкил этди.

Ҳисоблаш тўри тахминан 1,99 млн. та элементдан (тетраэдр ва призмалар) иборат бўлиб, курак сирти ва интерфейс соҳаларида маҳаллий майдалаштирилди (ўртача элемент сифати 0.679). Радиал жойлашувни баҳолаш учун r_{co} қиймати 21–33 mm оралиғида (2 mm кадам билан) параметрик таҳлил қилинди.

III. Натижалар

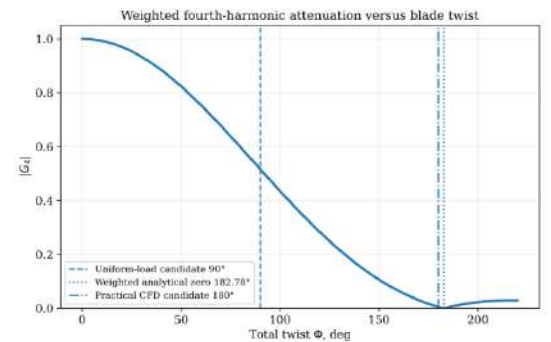
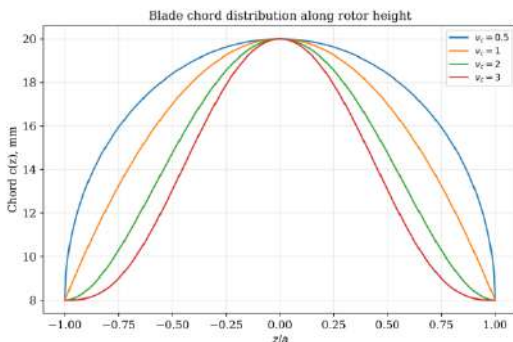
Эллиптик ташқи габарит параметрларини аналитик оптималлаштириш мақсад функциясида аниқ экстремум мавжудлигини кўрсатди (1-расм, а). Изланишлар натижасида ротор ярим ўқларининг мақбул нисбатлари $a/R_p = 1$ ва $b/R_p = 0.68804169$ ($a/b = 1.4534003$) эканлиги аниқланиб, мақсад функцияси қиймати $J_{opt} = 16.8058$ ни ташкил этди. Бу кўрсаткич сферик базавий ротор параметрига ($J_{sph} = 15$) нисбатан келтирилган момент потенциаллини 12.04 % га оширишини исботлайди. Олинган натижа роторни радиал йўналишда кенгайтиришдан кўра, уни қувур ўқи бўйлаб чўзиш орқали марказий юқори тезликли оқим ҳудудидан тўлиқроқ фойдаланиш мақбулликдан далолат беради. Шу билан бирга, куракнинг радиал жойлашуви (r_{co}) бўйича гидродинамик ва геометрик чегаралар ўртасидаги мутаносиблик баҳоланди (1-расм, б). Қувурдаги турбулент тезлик профили $m_v = 7$ учун назарий гидродинамик максимум $r_{co,hyd} = 38.89$ mm да ҳосил бўлса-да, бу қиймат эллиптик ташқи чегарадан ($b = 34.40$ mm) ташқарида жойлашади. Бироқ эллиптик чегара соҳасида гидродинамик момент

потенциалининг 97.46% и сақланиб қолиши, яъни геометрик чеклов туфайли йўқотиш атиги 2.54 % ни ташкил этиши исботланди.



1-Расм. а -Эллиптик ташиқи габаритнинг аналитик оптималлаштириши харитаси. Юлдузча $a/R_p = 1$, $b/R_p = 0.68804$ нуктани кўрсатади; б- Курак радиал жойлашуви учун гидродинамик target ва эллиптик геометрик чеклов.

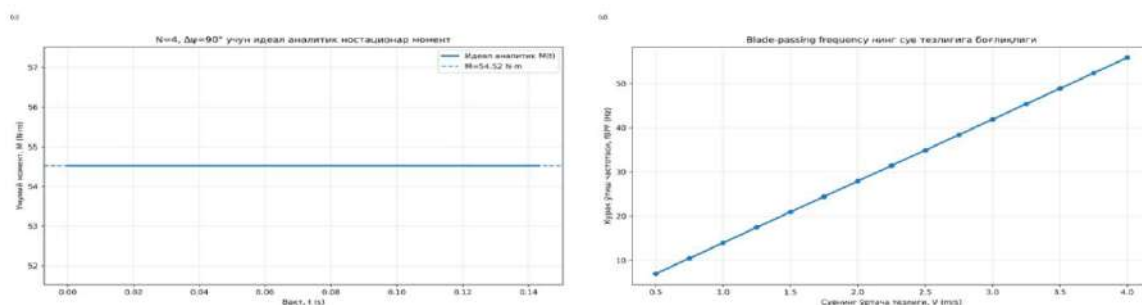
Курак ватарсининг баландлик бўйича тақсмоти $c(z)$ марказий қисмда кенгайиб, четларида торайиши назарий жиҳатдан асосланди (2-расм, а). Чунки ротор баландлигининг $|z| \leq 0.5a$ соҳаси умумий келтирилган момент потенциалининг 82.95% ини, $|z| \leq 0.75a$ эса 98.01% ини



2-расм.а) $c_h = 8$ mm ва $c_m = 20$ mm бўлганда v_c нинг турли қийматлари учун курак ватарсининг баландлик бўйича тақсмоти. б)Вазнланган тўртинчи гармоника амплитудасининг умумий фазовий буралиш Φ га боғлиқлиги.

Оптимал айланма тезлик коэффиценти $\lambda^* = 1$ сақланганда, бурчак тезлик $\omega_{opt} = V/R_{eff}$ муносабати асосида оқим тезлиги билан чизиқли равишда ўсиб боради. $V = 2$ m/s да $\omega_{opt} = 43.848$ rad/s, $V = 4$ m/s да эса тахминан 87.7 rad/s ни ташкил этади. Босим фарқи ўзгармас бўлганда, назарий гидравлик ва вал қуввати оқим тезлигига мутаносиб равишда чизиқли ортиши кузатилади. Ўртача айлантирувчи момент эса асосан босим фарқи Δp билан масштабланди

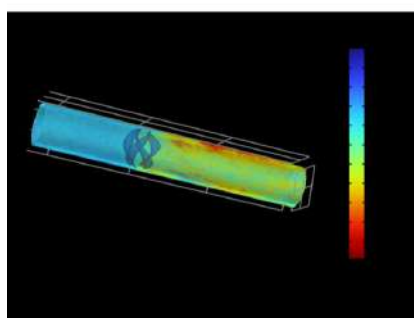
ва оқим тезлигига нисбатан деярли ўзгармас сақланди (3-расм, б). Номинал иш режимида ($\Delta p = 0.20$ МПа ҳолатда $V = 2$ m/s) аналитик модель бўйича $\bar{M} = 54.52$ N·m, $P_{shaft} = 2390.8$ W ва $\eta_h = 0.761$ кўрсаткичлари қайд этилди (3-расм, а, б, в, г). Мазкур тўрт куракли ротор ($Z_b = 4$) учун номинал айланиш частотасида курак ўтиш частотаси $f_{BPF} = Z_b \omega / (2\pi)$ бўлиб, 27.915 Гц ни ташкил этди.



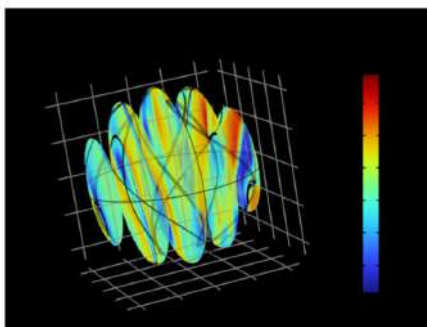
3-расм. Аналитик энергетик характеристикалар: (a) $\omega_{opt}(V)$; (b) $\bar{M}(V, \Delta p)$; (c) $P_{shaft}(V, \Delta p)$; (d) $\Delta p = 0.20$ МПа да гидравлик–вал–электр энергетик баланси.

COMSOL Multiphysics муҳитида олинган уч ўлчамли оқим майдонлари назарий таҳлил хулосаларини тўлиқ тасдиқлади (4-расм). Статик босим тақсимоти ротор олдида юқори босим соҳаси (6.08×10^5 Па), кураклар сиртида кескин босим градиенти ҳамда ротор ортида босимнинг пасайишини кўрсатди. Тезлик майдонида кураклар оралиғи ва чекка тирқишларда оқимнинг

маҳаллий тезланиши, ротор ортида эсано текис гирдобли из соҳаси намоён бўлди. Куракнинг радиал жойлашуви бўйича $r_{c0} = 21\text{--}33\text{ mm}$ оралиғида ўтказилган параметрик таҳлил куракни рухсат этилган ташқи чегарага яқинлаштириш гидравлик энергияни ажратиб олиш самарадорлигини оширишини намоёниш этди.



a)



b)

4-расм. (a) COMSOL CFD муҳитидаги оқим майдонлари: босим ва тезлик катталиги тақсимотлари ҳамда ротор ортидаги из соҳаси тузилиши; (b) COMSOL 3D модели: ҳисоблаш геометрияси, туташув интерфейси ва курак атрофидаги маҳаллий тўр майдаланиши.

Parametric Sweep $r_{c0} = 21, 23, 25, 27, 29, 31, 33$ mm бўйича ўтказилган параметрик таҳлил аналитик саралаш натижалари билан сонли гидродинамик (CFD) модел геометрияси ўртасидаги узвий боғлиқликни таъминлади.

IV. Муҳокама

Таклиф этилган ёндашувнинг концептуал афзаллиги геометрик ва энергетик кўрсаткичларни ўзаро боғлиқ чекловчи тизим сифатида интеграциялашдадир. Қувур ичига ўрнатиладиган анъанавий сферик ва кесишган оқимли микротурбиналарда кўндаланг кесимнинг тўсилиш даражаси (blockage) ортиши қувват коэффициентини оширса-да, босимнинг

кескин йўқотилишига олиб келиши кўрсатилган [7–12]. Мазкур тадқиқотда ушбу зиддият эллиптик ташқи габарит нисбатлари, девор оралиғидаги тирқиш ва курак марказининг радиал жойлашувини биргаликда оптималлаштириш орқали бартараф этилди. Натижада қайд этилган 12.04 % лик назарий момент устунлиги роторни радиал кенгайтириш эмас, балки уни ўқ бўйлаб чўзиш орқали марказий юқори кинетик энергияли оқимдан тўлиқ фойдаланиш ҳисобига таъминланди.

Куракнинг радиал жойлашуви r_{c0} таҳлили муҳим гидродинамик қонуниятни кўрсатди: тезлик профили бўйича назарий максимум 38.89 мм да ҳосил бўлса-да,

конструкциянинг эллиптик чегараси 34.40 мм ни ташкил этади. Бироқ геометрик чегара соҳасида назарий момент потенциалнинг 97.46 % и сақланиб қолиши кичик гидродинамик компромисс (2.54%) эвазига қувур девори билан ишончли функционал тирқишни сақлаш мумкинлигини тасдиқлайди. Бу хулоса Saini ва Saini [10] томонидан аниқланган «тирқиш–тўсилиш» мувозанати тамойилини куракнинг фазовий чизиги даражасига аниқлаштиради. Шунингдек, курак ватарсини баландлик бўйича ўзгартириш қонуни (2) марказий фаол ҳудудга кўпроқ юза ажратиш имконини бериб, Hasanzadeh ва ҳаммуаллифлар [13] ҳамда Mrigua ва ҳаммуаллифлар [14] ишларидаги сирт шаклининг таъсирини 3D ротор конструкциясига муваффақиятли татбиқ этади.

Момент пульсацияси бўйича олинган назарий ечимлар алоҳида аҳамиятга эга. Тўрт куракли роторда бир текис юкланишдаги классик 90° буралиш бурчаги баландлик бўйича нотекис юкланиш шароитида гармоникани тўлиқ сўндира олмайди ($|G_4| = 0.5162$). Бессель функцияси орқали вазнланган таҳлил биринчи сўниш нуктасини $\Phi_{opt} \approx 182.78^\circ$ га силжишини кўрсатди ва амалий $\Phi = 180^\circ$ бурчагининг ($|G_4| = 0.00563$) гидродинамик асосланганлигини исботлади. Номинал кўрсаткичлар $P_{shaft} = 2390.8$ $\eta_h = 76.1\%$ босимли тармоқларда маҳаллий дросселлаш ўрнига ихчам эллиптик агрегатлар орқали энергияни самарали рекуперация қилиш имкониятини очиб беради ва анъанавий PaT ҳамда саноат прототипларини курак геометрияси даражасидаги кўп мезонли дизайн билан бойитади.

V. Хулоса

Босимли сув узатиш қувурларида технологик талабдан ортиқча босим энергиясини самарали рекуперация қилиш мақсадида фазовий буралган эллиптик куракли микрогидротурбина роторининг аналитик ва CFD кўп мезонли лойиҳалаш методологияси асосланди ҳамда қуйидаги асосий натижаларга эришилди:

1.Геометрик нисбатлар ва энергетик устунлик: Эллиптик ротор ярим ўқларининг мақбул нисбатлари $a/R_p = 1$, $b/R_p = 0.68804$ ва $a/b = 1.45340$ эканлиги аниқланиб, роторни ўқ бўйлаб чўзиш ҳисобига сферик базавий кўринишга нисбатан келтирилган момент потенциали 12.04% га оширилди.

2.Гидродинамик ва конструктив мувозанат: Турбулент оқимдаги назарий радиал target($r_{c0,hyd} = 38.89\text{ mm}$) эллиптик габаритдан ($b = 34.40\text{ mm}$) ташқарида жойлашса-да, чегара зонасида момент потенциалнинг 97.46% и сақланиб қолиши исботланди ҳамда ватарнинг марказда кенгаювчи тақсимооти асосланди.

3.Пульсация сўниши ва гармоник модел: Тўрт куракли роторда баландлик бўйича нотекис юкланиш ҳисобга олинганда тўртинчи гармониканинг вазнланган биринчи ноли $\Phi_{opt} \approx 182.78^\circ$ да вужудга келиши аниқланди ва амалий CAD/CFD геометрияси сифатида $\Phi = 180^\circ$ танланди ($|G_4| = 0.00563$).

4.Энергетик кўрсаткичлар ва сонли текширув: Номинал режимда $V = 2.0\text{ m/s}$ ва $\Delta p = 0.20\text{ MPa}$, $\omega_{opt} = 43.848\text{ rad/s}$, назарий вал қуввати $P_{shaft} = 2390.8\text{ Wt}$, айлантирувчи момент $\dot{M} = 54.52\text{ N}\cdot\text{m}$ ва гидравлик ФИК $\eta_h = 0.761$ таъминланиб, 3D CFD майдонлари назарий гипотезаларни тўлиқ тасдиқлади.

Фойдаланилган адабиётлар

- [1]. Boroomandnia A., Rismanchi B., Wu W. Шаҳар ҳудудларида кичик гидроэнергетик тизимлар: имкониятлар ва муаммолар. *Renewable and Sustainable Energy Reviews*. 2022;169:112866. DOI: 10.1016/j.rser.2022.112866.
- [2]. Giudicianni C., Mitrovic D., Wu W., Ferrarese G., Pugliese F., Fernández-García I., Campisano A., De Paola F., Malavasi S., Maier H.R., Savic D., Creaco E. Сув тақсимлаш тармоқларида энергияни қайта олиш стратегиялари: адабиётлар шарҳи ва углерод чиқиндиларини камайтириш шароитидаги истиқболли йўналишлар. *Urban Water Journal*. 2024;21(10):1185–1200. DOI: 10.1080/1573062X.2023.2212271.
- [3]. Hamlehdar M., Yousefi H., Noorollahi Y., Mohammadi M. Кичик гидроэнергетика ёрдамида сув тақсимлаш тармоқларидан энергияни қайта олиш: Эрон мисоли. *Energy*. 2022;252:124024. DOI: 10.1016/j.energy.2022.124024.
- [4]. Stefanizzi M., Capurso T., Balacco G., Binetti M., Camporeale S.M., Torresi M. Сув тақсимлаш тармоқларида турбина режимида ишловчи насосларни танлаш, бошқариш ва техник-иқтисодий мақбуллигини баҳолаш. *Renewable Energy*.

- 2020;162:1292–1306. DOI: 10.1016/j.renene.2020.08.108.
- [5]. Mitrovic D., Rodríguez Díaz J.A., García Morillo J., McNabola A. Сув тақсимлаш тармоқларида турбина режимида ишловчи насосларни танлашнинг оптималлаштиришга асосланган усули: мақсадлар ва ишлаш чегараларининг энг юқори самарадорлик нуктасига таъсири. *Journal of Water Resources Planning and Management*. 2021;147(5):04021019. DOI: 10.1061/(ASCE)WR.1943-5452.0001356.
- [6]. Kostner M.K., Zanfei A., Alberizzi J.C., Renzi M., Righetti M., Menarase A. Сув тақсимлаш тармоқларида турбина режимида ишловчи насосларни мақбул ўлчамлаш ва бошқариш орқали кичик гидроэнергия ишлаб чиқариш. *Applied Energy*. 2023;351:121802. DOI: 10.1016/j.apenergy.2023.121802.
- [7]. Muratoglu A., Demir M.S. Қувур ичида энергияни ўзгартириш учун сферик турбиналарни моделлаштириш. *Ocean Engineering*. 2022;246:110497. DOI: 10.1016/j.oceaneng.2021.110497.
- [8]. Kumar R., Sarkar S. Қувур ичи гидроэнергетик тизими учун сферик шакли Дарье турбинасининг иш кўрсаткичларини таҳлил қилиш. *Energy Conversion and Management*. 2023;294:117600. DOI: 10.1016/j.enconman.2023.117600.
- [9]. Murai Y., Umemura T., Park H.J., Horimoto Y., Tasaka Y. Қувур оқимида ишлайдиган сферик Савониус турбиналарининг тажрибавий ва назарий гидродинамикаси. *Journal of Fluids and Structures*. 2024;127:104105. DOI: 10.1016/j.jfluidstructs.2024.104105.
- [10]. Saini G., Saini R.P. Аралаш турдаги гидрокинетик турбина гидродинамик самарадорлигига тирқиш ва оқим кесимининг тўсилиш таъсири. *Sustainable Energy Technologies and Assessments*. 2023;57:103210. DOI: 10.1016/j.seta.2023.103210.
- [11]. Du X., Tan J., Yuan P., Si X., Liu Y., Wang S. Оқимни кенгайтирувчи қурилмали гидрокинетик турбинада тўсилиш таъсирини тузатишни тадқиқ этиш. *Ocean Engineering*. 2023;280:114470. DOI: 10.1016/j.oceaneng.2023.114470.
- [12]. Zhang D., Guo P., Cheng Y., Hu Q., Li J. Кураклар билан кўп тўсилган гидрокинетик турбиналарда тўсилиш таъсирини тузатиш усуллари: тажрибавий ва сонли тадқиқотлар. *Ocean Engineering*. 2023;283:115185. DOI: 10.1016/j.oceaneng.2023.115185.
- [13]. Hasanzadeh N., Najafi A.F., Riasi A. Қувур ичида ишлайдиган қаршилиқ кучига асосланган гидротурбиналарда курак турининг энергия олиш самарадорлигига таъсири. *Energy Conversion and Management*. 2024;314:118722. DOI: 10.1016/j.enconman.2024.118722.
- [14]. Mrigua K., Toumi A., Zemamou M., Ouhmmou B., Lahlou Y., Aggour M. Эллиптик куракли янги кўп босқичли Савониус роторларини ҳисоблаш гидродинамикаси асосида тадқиқ этиш. *International Journal of Renewable Energy Development*. 2020;9(3):383–392. DOI: 10.14710/ijred.2020.30286.
- [15]. Itani Y., Soliman M.R., Kahil M. Сув узатиш қувурларида гидротурбиналардан фойдаланиб энергияни қайта олиш: Саудия Арабистони ғарби мисоли. *Energy*. 2020;211:118613. DOI: 10.1016/j.energy.2020.118613.
- [16]. Hannachi M., Ketata A., Sinagra M., Aricò C., Tucciarelli T., Driss Z. Сарфнинг ҳисобий ва ҳисобдан ташқари қийматларида энергияни қайта олиш учун Банки гидротурбинасига асосланган янги босим ростлаш тизими. *Energy Conversion and Management*. 2021;243:114417. DOI: 10.1016/j.enconman.2021.114417.
- [17]. Mahauddin H.A., Jauhari I., Nik Ghazali N.N., Mohd Sultan N., Julai S. Қувур ичида ишлайдиган янги гидроэнергетик қурилма намунасини ишлаб чиқиш ва тайёрликнинг олтинчи даражасигача синаш. *Cogent Engineering*. 2024;11(1):2319402. DOI: 10.1080/23311916.2024.2319402.
- [18]. Bonthuys G.J., van Dijk M., Cavazzini G. Гидротурбиналар ёрдамида сув тақсимлаш тизимларида энергияни қайта олиш ва сув сизиб чиқишини камайтиришни оптималлаштириш. *Journal of Water Resources Planning and Management*. 2020;146(5):04020026. DOI: 10.1061/(ASCE)WR.1943-5452.0001203.
- [19]. Balacco G., Fiorese G.D., Alfio M.R., Totaro V., Binetti M., Torresi M., Stefanizzi M. Сув тақсимлаш тармоғи учун турбина режимида ишловчи мақбул насосни танлаш воситаси. *Energy*. 2023;282:128366. DOI: 10.1016/j.energy.2023.128366.
- [20]. Manservigi L., Venturini M., Losi E., Castorino G.A.M. Энергияни қайта олишни максималлаштириш мақсадида турбина режимида ишловчи насосларни мақбул танлаш ва ишлатиш. *Water*. 2023;15(23):4123. DOI: 10.3390/w15234123.
- [21]. Le Marre M., Mandin P., Lanoisellé J.-L., Zilliox E., Rammal F., Kim M., Inguanta R. Турбина режимида ишловчи насосларни қарор қабул қилиш алгоритми ёрдамида ростлашни тадқиқ этиш. *Renewable Energy*. 2022;194:561–570. DOI: 10.1016/j.renene.2022.05.128.
- [22]. Corrêa S.S., Souza J.D.S., Oliveira A.A.A., Bezerra S.T.M. Сув тақсимлаш тизимларида энергияни қайта олиш учун турбина режимида ишловчи насослар: шарҳ. *Proceedings of the Institution of Civil Engineers – Energy*. 2024;177(2):74–84. DOI: 10.1680/jener.23.00001.
- [23]. Abdullah M.F., Jauhari I., Mohd Sabri M.F., Nik Ghazali N.N. Қувур ичида гидроэнергия ишлаб чиқариш учун параллел вертикал ўқли янги турбиналар тизими: конструктив ғоя ва дастлабки тажриба. *Energy Sources, Part A: Recovery, Utilization, and Environmental Effects*. 2025;47(1):4722–4736. Published online 8 February 2021. DOI: 10.1080/15567036.2021.1880501.

Turli sintez usullarida olingan ZnO materiallarining optik va fotoluminessensiya xossalari qiyosiy tahlili.

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Annotatsiya. Ushbu ishda oksidlanish–qaytarilish va zol-gel usullarida sintez qilingan ZnO materiallarining optik hamda fotoluminessensiya xossalari qiyosiy o'rganildi. Fotoluminessensiya spektrlari tahlili asosida ultrabinafsha va ko'zga ko'rinuvchi diapazonlardagi emissiyaning kristall panjara nuqsonlari va eksiton rekombinatsiyasi bilan bog'liqligi aniqlandi. Olingan natijalar oksidlanish–qaytarilish usulida tayyorlangan namunalarning yuqori kristall mukammallikka ega ekanligini, zol-gel usulida olingan ZnO qatlamlarida esa interfeys nuqsonlari bilan bog'liq qo'shimcha emissiya kuzatilishini ko'rsatdi. Tadqiqot natijalari ZnO asosidagi materiallarni ultrabinafsha diapazonida ishlovchi optoelektron qurilmalar uchun istiqbolli material sifatida qo'llash imkoniyatlarini tasdiqlaydi.

Kalit so'zlar: rux oksidi (ZnO), fotoluminessensiya, oksidlanish–qaytarilish usuli, zol-gel, eksiton rekombinatsiyasi, optoelektronika.

Аннотация. В данной работе проведено сравнительное исследование оптических и фотолюминесцентных свойств материалов ZnO, синтезированных методами окислительно-восстановительной реакции и sol–gel. Анализ спектров фотолюминесценции показал, что ультрафиолетовая и видимая области излучения связаны с дефектами кристаллической решётки и рекомбинацией экситонов. Полученные результаты свидетельствуют о том, что образцы, полученные методом окислительно-восстановительной реакции, обладают более высоким кристаллическим совершенством, тогда как в слоях ZnO, синтезированных методом sol–gel, наблюдается дополнительная эмиссия, обусловленная межфазными дефектами. Полученные результаты подтверждают перспективность материалов на основе ZnO для применения в ультрафиолетовых оптоэлектронных устройствах.

Ключевые слова: оксид цинка (ZnO), фотолюминесценция, окислительно-восстановительный метод, метод sol–gel, рекомбинация экситонов, оптоэлектроника.

Abstract. This study presents a comparative investigation of the optical and photoluminescence properties of ZnO materials synthesized by redox and sol–gel methods. The analysis of the photoluminescence spectra revealed that the ultraviolet and visible emission bands are associated with crystal lattice defects and exciton recombination. The results demonstrate that the samples prepared by the redox method exhibit higher crystalline quality, whereas the ZnO layers synthesized by the sol–gel method show additional emission related to interface defects. The obtained findings confirm the potential of ZnO-based materials for applications in ultraviolet optoelectronic devices.

Keywords: zinc oxide (ZnO), photoluminescence, redox method, sol–gel, exciton recombination, optoelectronics.

I. Kirish.

Bugungi kunda ultrabinafsha (UB) diapazonda ishlovchi fotoqabul qilgichlar, yorug'lik nurlanuvchi qurilmalar va boshqa optoelektron tizimlarni yaratishda keng taqiqlangan zonali yarimo'tkazgich materiallarga bo'lgan qiziqish ortib bormoqda. Bunday materiallar qatoriga GaN, SiC va ZnO kiradi. Ular orasida rux oksidi (ZnO) taxminan 3,3 eV taqiqlangan zona kengligi hamda 60 meV eksiton bog'lanish energiyasiga ega bo'lib, yuqori optik shaffofligi, kimyoviy barqarorligi va nisbatan arzon texnologik jarayonlari tufayli istiqbolli material hisoblanadi[1].

ZnO materiallarining optik xossalari, ayniqsa fotoluminessensiya spektrlari, ularning kristall mukammalligi va nuqtaviy

nuqsonlar konsentratsiyasiga bevosita bog'liq. Fotoluminessensiya spektrida odatda 380 nm atrofida ultrabinafsha emissiya erkin eksitonlarning rekombinatsiyasi bilan bog'liq bo'lsa, 450–550 nm oralig'idagi ko'zga ko'rinuvchi emissiya kislorod vakansiyalari, rux kirishmalari hamda boshqa nuqson sathlari ishtirokidagi rekombinatsion jarayonlar natijasida hosil bo'ladi[2]. Shu sababli fotoluminessensiya spektrlarini tahlil qilish ZnO materiallarining strukturaviy mukammalligi va optik sifatini baholashning samarali usullaridan biri hisoblanadi.[3]

ZnO qatlamlarini olishda oksidlanish–qaytarilish, zol-gel, magnetron purkash, kimyoviy bug' fazasidan o'stirish va boshqa texnologiyalar qo'llaniladi. Mazkur usullar olingan materiallarning kristall tuzilishi, nuqsonlar zichligi hamda optik xossalari

turlicha ta'sir ko'rsatadi [4]. Shu bois sintez usulining ZnO materiallarining fotoluminessensiya va optik xossalarga ta'sirini qiyosiy o'rganish ilmiy va amaliy jihatdan dolzarb masala hisoblanadi.[7]

Mazkur ishning maqsadi oksidlanish–qaytarilish va zol-gel usullarida sintez qilingan ZnO materiallarining optik hamda fotoluminessensiya xossalarni qiyosiy tahlil qilish, ultrabinafsha va ko'zga ko'rinuvchi diapazonlardagi emissiya xususiyatlarini aniqlash hamda sintez usulining kristall mukammalligi va optik xossalarga ta'sirini baholashdan iborat.

II. Tadqiqot metodikasi.

Yarimo'tkazgich materiallarining optik xossalari yorug'likning material bilan o'zaro ta'siri natijasida yuzaga keladigan yutilish, qaytish va o'tkazish jarayonlari orqali tavsiflanadi. Kristall sirtiga tushayotgan yorug'lik intensivligi quyidagi energiya balansini ifodalaydi:

$$I = I_q + I_T + I_a \quad (1)$$

bu yerda I – tushayotgan, I_q – qaytgan, I_T – o'tgan va I_a – yutilgan yorug'lik intensivliklaridir.

Yorug'likning material ichida yutilishi Buger–Lambert qonuni bilan tavsiflanadi[6]:

$$I = I_0(1 - q) \cdot \exp(\alpha d) \quad (2)$$

bu yerda I_0 – boshlang'ich yorug'lik intensivligi, d – namuna qalinligi, α – yutilish koeffitsienti. Ushbu qonun yorug'lik intensivligining material qalinligi ortishi bilan eksponensial ravishda kamayishini ifodalaydi. Yutilish koeffitsienti asosida yarimo'tkazgichning ta'qiqlangan soha kengligi T_{auc} munosabati yordamida aniqlanadi [7]:

$$(\alpha h\nu)^2 = A(h\nu - E_g) \quad (3)$$

Bu yerda, E_g – ta'qiqlangan soha kengligi; h – Plank doimiysi; ν – tebranish chastotasi; α – yutilish koeffitsienti; A – proporsionallik koeffitsienti.

ZnO keng ta'qiqlangan zonali ($E_g \approx 3.34$ eV) yarimo'tkazgich bo'lib, yuqori eksiton bog'lanish energiyasi (≈ 60 meV) bilan ajralib turadi. Shu sababli xona haroratida ham kuchli ultrabinafsha fotoluminessensiya kuzatiladi. Fotoluminessensiya spektridagi 380 nm atrofidagi tor emissiya cho'qqisi erkin

eksitonlarning rekombinatsiyasi bilan bog'liq bo'lib, materialning yuqori kristall mukammalligini tavsiflaydi.

Ko'zga ko'rinuvchi diapazondagi keng polosali emissiya esa, asosan, kristall panjaradagi nuqtaviy nuqsonlar bilan bog'liq rekombinatsion jarayonlar natijasida yuzaga keladi[2,3]. Xususan, kislorod vakansiyalari (V_O), rux kirishmalari (Zn_i) hamda donor–akseptor sathlari ishtirokidagi o'tishlar 450–550 nm oralig'idagi yashil emissiyaning hosil bo'lishiga sabab bo'ladi. Shuning uchun fotoluminessensiya spektrlarining shakli va intensivligi ZnO materiallarining kristall sifati hamda nuqsonlar konsentratsiyasini baholashning muhim mezonini hisoblanadi.

III. Natijalar va muhokama

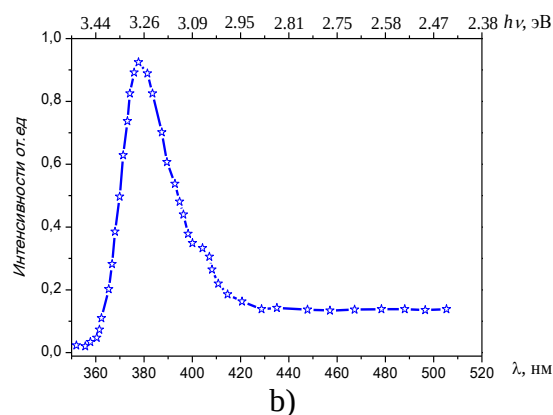
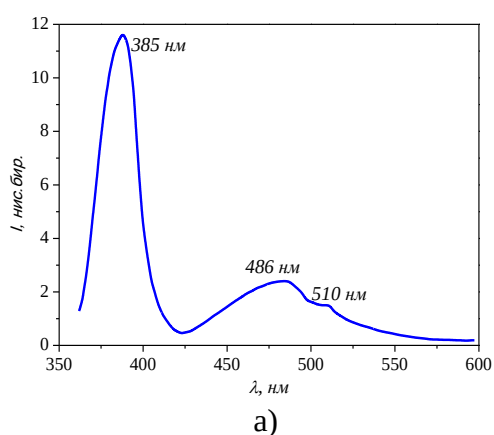
Oksidlanish–qaytarilish va zol-gel usullarida sintez qilingan ZnO materiallarining fotoluminessensiya spektrlari 1-rasmda keltirilgan. Har ikkala namuna ultrabinafsha diapazonda kuchli emissiya namoyon qilgan bo'lsa-da, spektrlarning shakli va intensivligida sezilarli farqlar kuzatiladi. Ushbu farqlar sintez usulining kristall mukammalligi, nuqsonlar konsentratsiyasi hamda rekombinatsion jarayonlarga ta'siri bilan izohlanadi. 1-rasm (a) dan ko'rinib turibdiki, oksidlanish–qaytarilish usulida olingan ZnO monokristalining fotoluminessensiya spektrida 385 nm atrofida intensiv ultrabinafsha emissiya kuzatiladi. Ushbu emissiya erkin eksitonlarning rekombinatsiyasi bilan bog'liq bo'lib, materialning yuqori kristall mukammalligini ko'rsatadi. Bundan tashqari, 486 nm va 510 nm oralig'ida kuzatilgan keng polosali emissiya kislorod vakansiyalari, rux interstitsiallari hamda donor–akseptor nuqson sathlari ishtirokidagi rekombinatsion jarayonlar bilan izohlanadi.

1-rasm (b) da oksidlanish–qaytarilish usuli yordamida sintez qilingan monokristall ZnO namunasining fotoluminessensiya spektri keltirilgan. Namunalar galliy ishtirokida vodorod atmosferasida oksidlanish–qaytarilish reaksiyasi orqali tayyorlangan. Sintez jarayonida taglik harorati 500–670 °C, manba sohasi esa 720–750 °C oralig'ida ushlab turilgan. Keyingi bosqichda namunalar kislorodli muhitda 700–750 °C da termik ishlav

berilib, so'ng vodorod atmosferasida 550 °C da qo'shimcha ushlab turiladi. Zol-gel texnologiyasi asosida tayyorlangan n-ZnO yupqa qatlamining fotoluminessensiya spektri keltirilgan. Spektrning asosiy maksimumi 377–380 nm oralig'ida joylashgan bo'lib, u ZnO ning fundamental eksiton rekombinatsiyasiga mos keladi. Shuningdek, 400–410 nm oralig'ida qo'shimcha emissiya kuzatilishi interfeys nuqsonlari hamda nanoo'lchamli strukturalarning hosil bo'lishi bilan bog'liq deb hisoblanadi. Spekrda 500–550 nm diapazonidagi nuqson emissiyasining deyarli kuzatilmasligi kislorod vakansiyalari

konsentratsiyasining kamayganligini va optik sifatning yaxshilanganligini ko'rsatadi[8].

Olingan natijalar shuni ko'rsatadiki, oksidlanish–qaytarilish usulida sintez qilingan ZnO namunalari yuqori kristall mukammalligi va kuchli eksiton emissiyasi bilan tavsiflanadi. Zol-gel texnologiyasi esa texnologik jihatdan sodda va iqtisodiy samarali bo'lishiga qaramay, interfeys nuqsonlari bilan bog'liq qo'shimcha emissiyaning paydo bo'lishiga olib keladi. Shu bilan birga, kislorod vakansiyalari bilan bog'liq yashil emissiyaning kamayishi optik sifatning yaxshilanganligini ko'rsatadi.



1-rasm. Turli sintez usullarida olingan ZnO materiallarining fotoluminessensiya spektrlari: (a) oksidlanish–qaytarilish usulida olingan ZnO monokristali; (b) zol-gel texnologiyasi asosida tayyorlangan n-ZnO yupqa qatlami.

III. Xulosa

Ushbu ishda oksidlanish–qaytarilish va zol-gel usullarida sintez qilingan ZnO materiallarining optik hamda fotoluminessensiya xossalari qiyosiy tahlil qilindi. Fotoluminessensiya spektrlari tahlili natijasida har ikkala usulda olingan namunalarda ultrabinafsha emissiya kuzatilishi ZnO ning yuqori optik faolligini tasdiqladi. Oksidlanish–qaytarilish usulida tayyorlangan namunalarda kuchli eksiton

emissiyasi va yuqori kristall mukammallik kuzatilgan bo'lsa, zol-gel usulida sintez qilingan ZnO qatlamlarida interfeys va nuqtaviy nuqsonlar bilan bog'liq qo'shimcha emissiya qayd etildi. Tadqiqot natijalari sintez usuli ZnO materiallarining optik va fotoluminessensiya xossalari sezilarli ta'sir ko'rsatishini hamda ularni ultrabinafsha diapazonida ishlovchi optoelektron qurilmalar uchun mos material sifatida qo'llash imkoniyatlarini tasdiqlaydi.

Thesis. Moscow Institute of Physics and Technology, Moscow, 2019.

[3]. Look D. C. *Recent advances in ZnO materials and devices*. Materials Science and Engineering B, 80, 383–387 (2001). [https://doi.org/10.1016/S0921-5107\(00\)00604-8](https://doi.org/10.1016/S0921-5107(00)00604-8)

[4]. Willander M., Nur O., Sadaf J. R., et al. *Zinc oxide nanorod based photonic devices: recent progress in growth, light emitting diodes and lasers*. Nanotechnology, 20, 332001 (2009). <https://doi.org/10.1088/0957-4484/20/33/332001>

Adabiyotlar

[1]. Özgür Ü., Alivov Ya. I., Liu C., et al. A comprehensive review of ZnO materials and devices. *Journal of Applied Physics*. 98(4), 041301 (2005). <https://doi.org/10.1063/1.1992666>

[2]. Tarasov A. P. Luminescence of zinc oxide microstructures and the influence of surface plasmon resonance and magnetic field. PhD

- [5]. Kołodziejczak-Radzimska A., Jesionowski T. *Zinc Oxide—From Synthesis to Application: A Review*. *Materials*, 7(4), 2833–2881 (2014). <https://doi.org/10.3390/ma7042833>
- [6]. Swanepoel R. Determination of the thickness and optical constants of amorphous silicon. *Journal of Physics E: Scientific Instruments*. 16, 1214–1222 (1983).
- [7]. Tauc J., Grigorovici R., Vancu A. Optical properties and electronic structure of amorphous germanium. *Physica Status Solidi*. **15**, 627–637 (1966).
- [8]. Burakov V. S. Morphology and optical properties of ZnO nanostructures synthesized by thermal and electrical discharge sputtering. *Technical Physics*. 81(2), 2011.

ULTRATOVUSHLI PURKASH PIROLIZI USULIDA OLINGAN ZNO VA ZNO:S:N YUPQA PLYONKALARINING OPTIK YUTILISH, O'TKAZUVCHANLIK VA URBACH ENERGIYASI TAHLILI

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Annotatsiya: Ushbu ishda shisha substratlar yuzasida ultratovushli purkash pirolizi (USP) usuli yordamida sof ZnO hamda S va N bilan modifikatsiyalangan ZnO:S:N yupqa plyonkalar sintez qilindi. Olingan namunalarning optik yutilish, optik o'tkazuvchanlik, taqiqlangan zona energiyasi va yutilish chekkasi yaqinidagi energetik dum holatlari UV–Vis spektroskopiyasi, Tauc va Urbach tahlillari asosida o'rganildi. Dastlab shisha substratning optik javobi baholanib, uning 400–550 nm diapazonda yuqori shaffoflikka ega ekanini aniqlandi; ushbu sohada optik o'tkazuvchanlik 91–92% ni tashkil etdi, Tauc tahlili orqali esa optik taqiqlangan zona energiyasi $E_g \approx 3.85\text{ eV}$ deb baholandi. Sof ZnO yupqa plyonkalarida ultrabinafsha sohada kuchli yutilish va ko'rinadigan sohada yuqori optik o'tkazuvchanlik kuzatildi. Absorbsiya chekkasi 370–390 nm oralig'ida shakllanib, Tauc grafigi asosida optik taqiqlangan zona energiyasi $E_g = 3.27\text{ eV}$ deb aniqlandi. Sof ZnO uchun Urbach energiyasining $E_u = 74.6\text{ meV}$ bo'lishi yutilish chekkasining nisbatan keskinligini va lokal nuqson holatlarining cheklanganligini ko'rsatadi. S va N bilan modifikatsiyalangan ZnO:S:N plyonkalarida esa ko'rinadigan sohadagi optik o'tkazuvchanlik 78–83% oralig'ida saqlangan bo'lsa-da, optik yutilish fonining ortishi va yutilish chekkasining past energiya tomonga siljishi qayd etildi. Ushbu namuna uchun $E_g = 3.17\text{ eV}$ gacha kamayishi va $E_u = 268\text{ meV}$ gacha ortishi S va N qo'shimchalari ta'sirida lokal energetik sathlar, strukturaviy tartibsizlik hamda Urbach dum holatlarining kuchayganini ko'rsatdi. Olingan natijalar ZnO asosidagi yupqa plyonkalarining optik parametrlarini USP texnologiyasi va modifikatsiyalash orqali boshqarish mumkinligini hamda bunday materiallarning shaffof optoelektronika, UV-sezgir funksional qoplamalar va keng taqiqlangan zonali yarimo'tkazgich qurilmalari uchun istiqbolli ekanini ko'rsatadi.

Kalit so'zlar: ZnO yupqa plyonkalar, ZnO:S:N, Ultratovushli purkash pirolizi (USP), Shisha substrat, UV–Vis spektroskopiya, Optik yutilish, Optik o'tkazuvchanlik, Tauc usuli, Optik taqiqlangan zona energiyasi (Band gap), Urbach energiyasi

Аннотация. В работе методом ультразвукового спрей-пиролиза (USP) на стеклянных подложках синтезированы тонкие плёнки чистого ZnO и ZnO:S:N, модифицированного серой и азотом. Оптическое поглощение, пропускание, ширина запрещённой зоны и энергетические хвостовые состояния вблизи края поглощения исследованы с помощью UV–Vis-спектроскопии и анализа Тауца и Урбаха. Стеклянная подложка показала высокую прозрачность в диапазоне 400–550 нм с пропусканием 91–92%; её оптическая ширина запрещённой зоны оценена как $E_g \approx 3.85\text{ эВ}$. Плёнки чистого ZnO характеризовались сильным поглощением в ультрафиолетовой и высоким пропусканием в видимой области. Край поглощения находился в диапазоне 370–390 нм; по графику Тауца определено $E_g = 3.27\text{ эВ}$. Энергия Урбаха $E_u = 74.6\text{ мэВ}$ указывает на относительно резкий край поглощения и ограниченное количество локальных дефектных состояний. В плёнках ZnO:S:N пропускание в видимой области сохранялось на уровне 78–83%, при этом наблюдались рост фона поглощения и смещение его края в область меньших энергий. Уменьшение E_g до 3.17 эВ и увеличение E_u до 268 мэВ указывают на усиление локальных энергетических состояний, структурного беспорядка и хвостовых состояний Урбаха под влиянием добавок S и N. Результаты показывают возможность управления оптическими параметрами плёнок на основе ZnO посредством технологии USP и модифицирования, а также перспективность таких материалов для прозрачной оптоэлектроники, УФ-чувствительных функциональных покрытий и широкозонных полупроводниковых устройств.

Ключевые слова: тонкие плёнки ZnO, ZnO:S:N, ультразвуковой спрей-пиролиз (USP), стеклянная подложка, UV–Vis-спектроскопия, оптическое поглощение, оптическое пропускание, метод Тауца, оптическая ширина запрещённой зоны, энергия Урбаха.

Abstract. Pristine ZnO and S- and N-modified ZnO:S:N thin films were synthesized on glass substrates using ultrasonic spray pyrolysis (USP). Optical absorption, transmittance, band gap energy, and energetic tail states near the absorption edge were investigated using UV–Vis spectroscopy and Tauc and Urbach analyses. The optical response of the glass substrate was first evaluated, revealing high transparency in the 400–550 nm range, with a transmittance of 91–92% and a Tauc-estimated optical band gap of $E_g \approx 3.85\text{ eV}$. Pristine ZnO films exhibited strong ultraviolet absorption and high visible transmittance. The absorption edge occurred at 370–390 nm, and the Tauc plot yielded $E_g = 3.27\text{ eV}$. The Urbach energy of $E_u = 74.6\text{ meV}$ indicates a relatively sharp absorption edge and limited local defect states. Although the

visible transmittance of ZnO:S:N films remained within 78–83%, an increased absorption background and a shift of the absorption edge toward lower energies were observed. The decrease in E_g to 3.17 eV and the increase in E_u to 268 meV indicate enhanced local energy states, structural disorder, and Urbach tail states induced by S and N additions. These results demonstrate that the optical parameters of ZnO-based thin films can be controlled through USP processing and modification, highlighting their potential for transparent optoelectronics, UV-sensitive functional coatings, and wide-band-gap semiconductor devices.

Keywords: ZnO thin films, ZnO:S:N, ultrasonic spray pyrolysis (USP), glass substrate, UV–Vis spectroscopy, optical absorption, optical transmittance, Tauc method, optical band gap energy, Urbach energy.

KIRISH

Rux oksidi (ZnO) keng taqiqlangan zonali yarimo‘tkazgich materiallardan biri bo‘lib, ko‘rinadigan sohada yuqori optik shaffofligi, ultrabinafsha diapazonda kuchli yutilishi, kimyoviy barqarorligi, nisbatan arzonligi hamda turli yupqa plyonka olish texnologiyalari bilan mosligi sababli katta ilmiy va amaliy qiziqish uyg‘otadi. Shu xossalari tufayli ZnO asosidagi yupqa plyonkalar shaffof elektronika, optoelektron qurilmalar, ultrabinafsha fotodetektorlar, quyosh elementlari, gaz sensorlari va funksional optik qoplamalarda keng qo‘llanilishi mumkin [1-3]. Ayniqsa, ZnO ning ko‘rinadigan sohada yuqori o‘tkazuvchanlikka va to‘g‘ridan-to‘g‘ri optik o‘tishga ega keng taqiqlangan zonaga ega bo‘lishi uni shaffof hamda UV-sezgir yarimo‘tkazgich strukturalar uchun istiqbolli materialga aylantiradi.

ZnO yupqa plyonkalarining fizik xususiyatlari ularni olish usuli, prekursor eritma tarkibi, substrat turi, cho‘ktirish harorati, plyonka qalinligi, kristallanish darajasi va nuqsonlar tizimiga kuchli bog‘liq. Real yupqa plyonkalarda fundamental yutilish chekkasi faqat ZnO ning ichki energetik zonalar tuzilishi bilan emas, balki kislorod vakansiyalari, rux interstitsiallari, donalararo chegaralar, sirt holatlari va boshqa strukturaviy nomukammalliklar bilan ham belgilanadi. Bunday nuqsonlar yutilish chekkasi yaqinida lokal energetik sathlar va energetik “dum” holatlarining shakllanishiga olib kelishi mumkin. Shu sababli ZnO asosidagi yupqa plyonkalarda optik yutilish, optik o‘tkazuvchanlik, taqiqlangan zona energiyasi va Urbach energiyasini birgalikda tahlil qilish texnologik sharoitlar, nuqsonlar tizimi va optik xossalari o‘rtasidagi bog‘liqlikni aniqlashda muhim ahamiyatga ega. ZnO ning optik va elektron tuzilmasini boshqarishning samarali usullaridan biri uni turli qo‘shimcha atomlar bilan modifikatsiyalashdir. Xususan, S va N kabi anion tabiatli qo‘shimchalar ZnO ning yutilish chekkasi, optik taqiqlangan zona kengligi va band chekkalari yaqinidagi lokal energetik holatlar zichligiga kuchli ta’sir ko‘rsatadi. S atomlarining ZnO panjarasiga kiritilishi kislorod atomlari bilan solishtirganda ion radiusi va bog‘lanish xususiyatlaridagi farqlar sababli panjara ichida lokal buzilishlar hosil qilishi

mumkin. N bilan modifikatsiyalash esa valensiya zonasi yaqinidagi holatlar shakllanishiga ta’sir ko‘rsatishi ehtimoli bilan ilmiy qiziqish uyg‘otadi [4-7]. Natijada S va N bilan modifikatsiyalangan ZnO yupqa plyonkalarida optik o‘tkazuvchanlik, yutilish foni, taqiqlangan zona energiyasi va Urbach dum holatlari sezilarli o‘zgarishi mumkin.

ZnO asosidagi yupqa plyonkalarni olishda magnetron changlatish, impulsli lazer cho‘ktirish, sol-gel, kimyoviy cho‘ktirish, atom qatlamlari cho‘ktirish va spray piroliz kabi usullar keng qo‘llaniladi. Ular orasida ultratovushli purkash pirolizi oddiyligi, iqtisodiy samaradorligi, katta yuzali substratlarda bir jinsli qoplama hosil qilish imkoniyati va prekursor tarkibini boshqarish qulayligi bilan ajralib turadi [8-9]. ZnO yupqa plyonkalarining optik xususiyatlarini baholashda UV–Vis spektroskopiyasi muhim ahamiyatga ega. Ushbu usul orqali yutilish chekkasi, ko‘rinadigan sohadagi optik o‘tkazuvchanlik, optik taqiqlangan zona energiyasi hamda yutilish chekkasi yaqinidagi lokal energetik holatlar haqida ma’lumot olish mumkin. ZnO to‘g‘ridan-to‘g‘ri ruxsat etilgan optik o‘tishli yarimo‘tkazgich bo‘lgani sababli uning band gap qiymatini aniqlashda Tauc usuli keng qo‘llaniladi. Urbach energiyasi esa plyonkadagi nuqsonlar, strukturaviy tartibsizlik va legirlash natijasida hosil bo‘ladigan energetik “dum” holatlarini baholash imkonini beradi.

Mazkur ishda sof ZnO hamda S va N bilan modifikatsiyalangan ZnO:S:N yupqa plyonkalarining optik xususiyatlari solishtirildi. S,N-modifikatsiyaning yutilish chekkasi, ko‘rinadigan sohadagi optik o‘tkazuvchanlik, optik taqiqlangan zona energiyasi va Urbach energiyasiga ta’sirini aniqlandi. sof ZnO va ZnO:S:N plyonkalarining Tauc hamda Urbach parametrlari asosida S,N-modifikatsiya natijasida optik band gapning o‘zgarishi va yutilish chekkasi yaqinidagi lokal energetik holatlarning kuchayishi baholandi.

MATERIALLAR VA METODLAR

Yupqa plyonkalarni o‘stirishda taglik sifatida shisha substratlardan foydalanildi. Cho‘ktirish jarayonidan oldin substratlar yuzasi iflosliklardan tozalash maqsadida atseton va etanol muhitida ultratovushli vannada 10 daqiqa davomida yuvildi. Shundan so‘ng ular distillangan suv bilan chayildi

va quritildi. Bunday tozalash jarayoni substrat yuzasida bir jinsli va yaxshi yopishuvchan plyonka hosil bo'lishini ta'minlaydi.

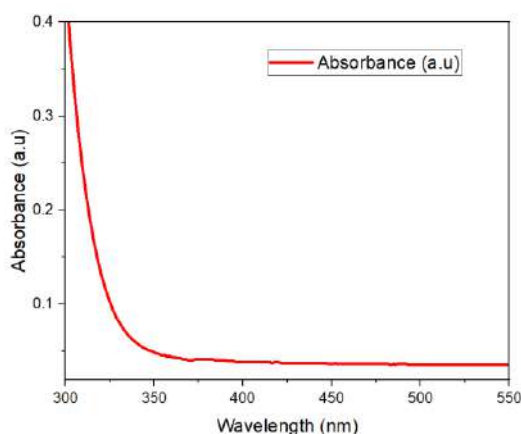
Prekursor eritma sifatida 0.1 M konsentratsiyali zinc acetate dihydrate $Zn(CH_3COO)_2 \cdot 2H_2O$ eritmasi tayyorlandi. S va N bilan modifikatsiyalangan $ZnO:S:N$ yupqa plyonkalarni olishda oltingugurt manbai sifatida tiomochevina $CS(NH_2)_2$, azot manbai sifatida esa ammoniy nitrat dihidrat $NH_4NO_3 \cdot 2H_2O$ qo'shimcha reagent sifatida ishlatildi. Sof ZnO hamda S,N-modifikatsiyalangan $ZnO:S:N$ yupqa plyonkalar ultratovushli purkash pirolizi usuli yordamida shisha substratlar yuzasida cho'ktirildi. Cho'ktirish jarayoni ishda keltirilgan texnologik parametrlar asosida amalga oshirildi. Cho'ktirish tugagach, namunalar xona haroratigacha tabiiy ravishda sovitildi. Plyonkalarining kristallanish darajasini yaxshilash maqsadida ular havo muhitida 400 °C haroratda 1 soat davomida termik ishlovdan o'tkazildi.

Olingan namunalar optik xususiyatlari UV–Vis spektroskopiyasi yordamida 300–800 nm to'lqin uzunliklari oralig'ida o'rganildi. Optik yutilish va o'tkazuvchanlik spektrlari asosida plyonkalarining optik taqiqlangan zona energiyasi (E_g) Tauc usuli yordamida aniqlandi. Yutilish chekkasi yaqinidagi lokal energetik holatlar esa Urbach energiyasi (E_u) orqali baholandi.

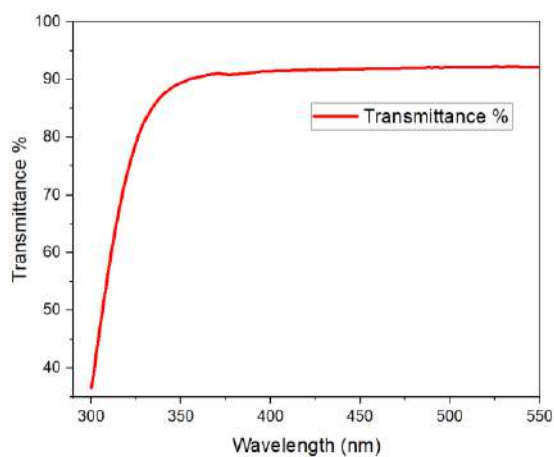
NATIJARLAR VA MUHOKAMA

Olingan optik yutilish va optik o'tkazuvchanlik spektrlari asosida namunalar uchun yutilish chekkasi, ko'rinadigan sohada shaffoflik darajasi, optik taqiqlangan zona energiyasi (E_g) hamda Urbach energiyasi (E_u) aniqlanib, ushbu parametrlar shisha substratning optik ta'sirini baholash, sof ZnO plyonkaning optik sifatini aniqlash va S hamda N bilan modifikatsiyalangan $ZnO:S:N$ plyonkalarida yuzaga kelgan energetik dum holatlarini tahlil qilish imkonini berdi.

Plyonka optik xususiyatlarini aniq baholash va uning taglikdan farqini aniqlash maqsadida dastlab faqat shisha substrat (taglik) o'rganildi. Chunki plyonka hosil bo'lishidan oldin taglikning optik xossalarini aniq baholab olinsa, plyonka qo'shilgandan keyin yuzaga keladigan o'zgarishlarni aniqroq baholash imkonini beradi. Absorbsiya spektridan ko'rinadiki shisha substrat ultrabinafsha sohada nisbatan kuchliroq yutilishga ega bo'lib (2a-rasm), 300 nm atrofida absorbsiya qiymati taxminan $A \approx 0.40$ ni tashkil etadi. To'lqin uzunligi ortishi bilan yutilish keskin kamayadi va 350–360 nm sohada $A \approx 0.04–0.05$ atrofida barqarorlashadi. 360 nm dan keyingi ko'rinadigan diapazonda absorbsiya juda kichik qiymatlarda saqlanib qoladi. Bu holat shisha substratning ko'rinadigan sohada deyarli yorug'likni yutmasligini va yuqori shaffoflikka ega ekanini ko'rsatadi.



a)



b)

2-rasm. Shisha substratning optik xususiyatlari. a) optik yutilish spektri; b) optik o'tkazuvchanlik spektri.

Optik o'tkazuvchanlik spektriga ko'ra, 300 nm yaqinida transmittans nisbatan past bo'lib, taxminan $T \approx 35–40\%$ ni tashkil etadi (2b-rasm). 300–350 nm oralig'ida transmittans tez ortadi va 350–370 nm atrofida $T \approx 88–91\%$ qiymatlarga yetdi. 380 nm dan boshlab 550 nm gacha bo'lgan ko'rinadigan sohada esa transmittans deyarli barqaror bo'lib, $T \approx 91–92\%$ oralig'ida saqlandi. Bu natija shisha substratning optoelektronika va yupqa plyonka

texnologiyalarida optik jihatdan qulay asos sifatida ishlatilishi mumkinligini ko'rsatadi.

Shisha substratning ko'rinadigan sohada transmittansi 100% ga yetmasligi tabiiy holat bo'lib, bu shisha–havo chegaralarida yorug'likning qaytishi, juda kichik sochilish yo'qotishlari hamda fon yutilish bilan bog'liq. Oddiy shisha uchun har bir sirt bo'yicha qaytish yo'qotishlari bir necha foizni tashkil etadi. Shu sababli ikki sirtli shisha

substratda umumiy optik o'tkazuvchanlikning 90–92% atrofida bo'ladi.

O'lchov natijalari asosida foton energiyasi quyidagi ifoda orqali hisoblandi:

$$h\nu = \frac{hc}{\lambda} = \frac{1240}{\lambda} \quad (1)$$

bu yerda $h\nu$ — foton energiyasi(eV), λ — to'lqin uzunligi, nm. Yutilish koeffitsienti α optik o'tkazuvchanlik orqali quyidagicha hisoblandi.

$$\alpha = \frac{1}{d} \ln \left(\frac{1}{T} \right) \quad (2)$$

bu yerda d — plyonka yoki substrat qalinligi, Tesa 0–1 oralig'idagi ulush. Agar optik o'lchovlar

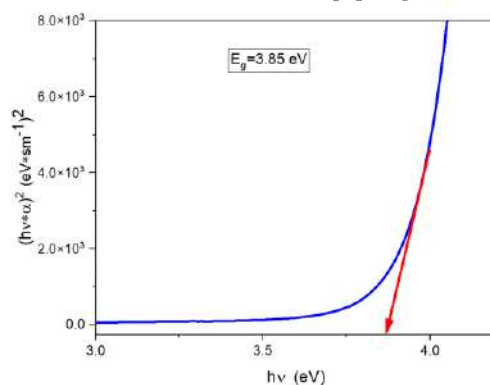
absorbsiya ko'rinishida berilgan bo'lsa, transmittans bilan bog'lanish quyidagicha ifodalanadi:

$$A = -\log_{10}(T), T = 10^{-A} \quad (3)$$

Optik taqiqlangan zona energiyasini aniqlash uchun Tauc usulidan foydalanildi. ZnO to'g'ridan-to'g'ri ruxsat etilgan optik o'tishli yarimo'tkazgich bo'lganligi sababli Tauc tenglamasi quyidagi ko'rinishda yozildi.

$$(\alpha h\nu)^2 = B(h\nu - E_g) \quad (4)$$

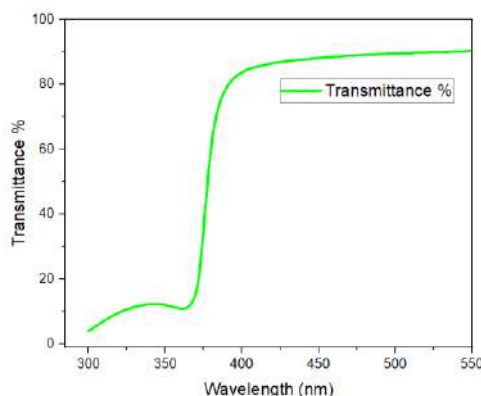
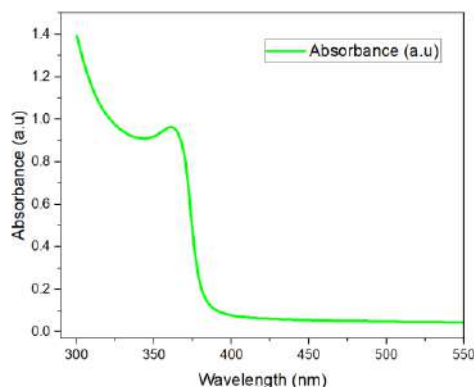
bu yerda B — doimiy koeffitsient, E_g — optik taqiqlangan zona energiyasi.



3-rasm. Tauc usuli yordamida shisha substratning optik taqiqlangan zona energiyasini (E_g) aniqlash.

Grafikdagi chiziqli qismni foton energiyasi o'qigacha ekstrapolyatsiya qilish orqali shisha substratning optik taqiqlangan zona energiyasi $E_g \approx 3.85$ eV deb baholandi (3-rasm). Ushbu qiymat shisha substratning keng taqiqlangan sohaga ega dielektrik material ekanini tasdiqlaydi.

Ultratovushli purkash pirolizi usuli yordamida shisha substrat yuzasida o'stirilgan sof ZnO yupqa plyonkasining optik yutilish va optik o'tkazuvchanlik spektrlari 4-rasmda keltirilgan. Grafiklardan ko'rinadiki, ZnO plyonkasi ultrabinafsha sohada kuchli yutilish, ko'rinadigan sohada esa yuqori optik shaffoflik namoyon etdi.



4-rasm. USP usuli bilan shisha substratda o'stirilgan sof ZnO yupqa plyonkasining optik xususiyatlari: a) optik yutilish spektri; b) optik o'tkazuvchanlik spektri.

300 nm atrofida yutilish eng yuqori bo'lib, $A \approx 1.35$ – 1.40 ni tashkil etdi (4a-rasm). To'lqin uzunligi ortishi bilan absorbsiya asta-sekin kamaydi va 340–360 nm oralig'ida nisbatan keng ko'rinishdagi soha kuzatildi. Ushbu hudud yutilish chekkasi yaqinidagi lokal energetik holatlar, sirt yoki interfeys ta'siri

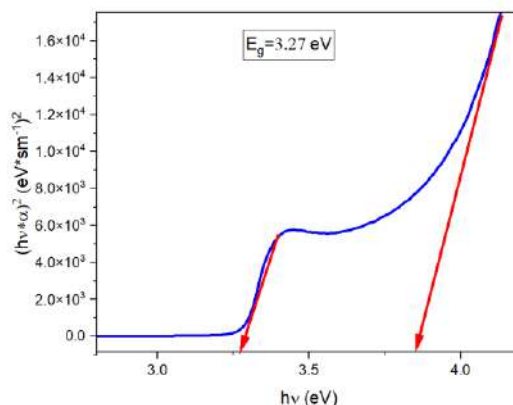
hamda kristall tuzilishdagi ayrim nuqsonlar bilan bog'liq bo'lishi mumkin [10].

Spektrning eng muhim qismi 370–390 nm oralig'ida kuzatiladigan keskin pasayish sohasidir. Aynan ushbu soha ZnO ning fundamental optik yutilish chekkasiga mos keladi. 400 nm dan keyingi ko'rinadigan diapazonda absorbsiya juda kichik

qiymatlarda, ya'ni $A \approx 0.04-0.07$ oralig'ida barqarorlashdi. Bu ZnO yupqa plyonkasining ko'rinadigan sohada yorug'likni kam yutishini va yuqori optik shaffoflikka ega ekanini ko'rsatadi.

300 nm da transmittans juda past bo'lib, taxminan $T \approx 4-6\%$ ni tashkil etdi (4b-rasm). 300–350 nm oralig'ida transmittans asta-sekin ortdi, 360–390 nm sohada esa keskin o'sish kuzatildi. 400

nmga yaqin hududda transmittans 75–85%gacha yetdi. 400–550 nm diapazonda esa ZnO plyonkasining optik o'tkazuvchanligi 88–91% oralig'ida deyarli barqaror saqlandi. Bu natija sof ZnO yupqa plyonkasining shaffof optoelektronik qatlamlar va UV-filtrlar uchun istiqbolli material ekanini ko'rsatadi[11].



5-rasm. Tauc usuli yordamida sof ZnO yupqa plyonkasining optik taqiqlangan zona energiyasini (E_g) aniqlash.

5-rasmda sof ZnO yupqa plyonkasi uchun $(\alpha h\nu)^2$ ning $h\nu$ ga bog'liqligi keltirilgan. Grafikdan ko'rinadiki, past energiyalar sohasida $(\alpha h\nu)^2$ qiymati kichik bo'lib, foton energiyasi ortishi bilan 3.2–3.4 eV oralig'ida keskin o'sish boshlandi. Ushbu chiziqli qism absissa o'qigacha ekstrapolyatsiya qilinganda, sof ZnO plyonkasi uchun optik taqiqlangan zona energiyasi $E_g \approx 3.27$ eV deb baholandi. Bu qiymat ZnO yupqa plyonkalari uchun xos bo'lgan 3.2–3.37 eV oralig'iga mos keladi va materialning keng taqiqlangan zonali ZnO yarimo'tkazgich tabiatini saqlab qolganini ko'rsatdi. Ideal yoki yuqori kristall sifatga ega ZnO uchun E_g qiymati odatda 3.30–3.37 eV atrofida bo'lishi kutiladi. Ushbu ishda esa E_g qiymatining 3.27 eV atrofida aniqlanishi o'stirish parametrlariga bog'liq bo'lib, bu plyonkadagi nuqson holatlari, yutilish chekkasi

yaqinidagi energetik dum holatlari va substrat ta'siri bilan bog'liq bo'lishi mumkin

ZnO yupqa plyonkasining yutilish chekkasi yaqinidagi tartibsizlik darajasini baholash uchun Urbach tahlili bajarildi. Urbach qonuniga ko'ra yutilish koeffitsienti quyidagicha ifodalanadi.

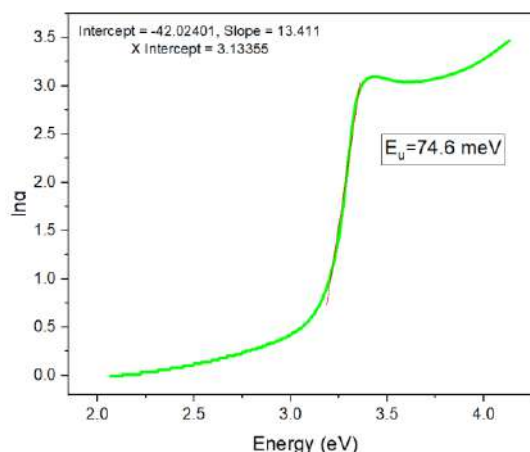
$$\alpha = \alpha_0 \exp\left(\frac{h\nu}{E_u}\right) \quad (5)$$

yoki logarifmik ko'rinishda:

$$\ln(\alpha) = \ln(\alpha_0) + \frac{h\nu}{E_u} \quad (6)$$

Demak, $\ln(\alpha)$ ning $h\nu$ ga bog'liqlik grafigidagi chiziqli qismning qiyaligi orqali Urbach energiyasi aniqlandi:

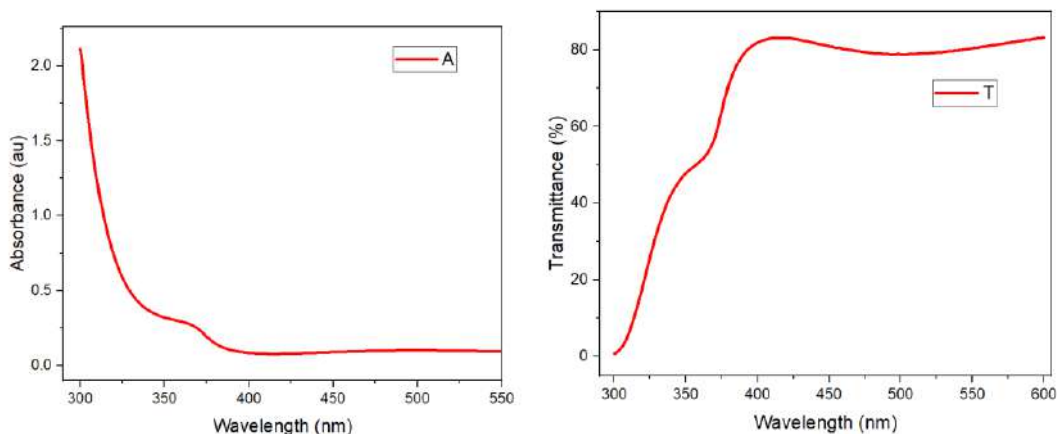
$$E_u = \left(\frac{d\ln(\alpha)}{d(h\nu)}\right)^{-1} \quad (7)$$



6-rasm. Sof ZnO yupqa plyonkasi uchun Urbach energiyasini (E_u) aniqlash grafigi.

6-rasm asosida sof ZnO yupqa plyonkasi uchun Urbach energiyasi $E_u = 74.6$ meV deb aniqlandi. Ushbu qiymat ZnO plyonkasida yutilish chekkasi nisbatan keskin shakllanganini, strukturaviy tartibsizlik va lokal nuqson holatlari cheklangan darajada ekanini ko'rsatdi.

S va N bilan modifikatsiyalangan ZnO:S:N yupqa plyonkasining optik yutilish va optik o'tkazuvchanlik spektrlari 7-rasmda keltirilgan. Grafiklardan ko'rinadiki, ZnO:S:N plyonkasi ham ultrabinafsha sohada kuchli yutilishga ega, biroq ko'rinadigan sohadagi optik o'tkazuvchanligi sof ZnO ga nisbatan biroz pastroq.



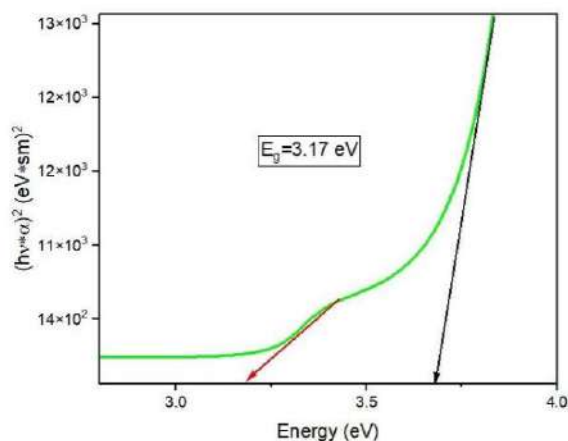
7-rasm. USP usuli bilan shisha substratda o'stirilgan ZnO:S:N yupqa plyonkasining optik xususiyatlari: a) optik yutilish spektri; b) optik o'tkazuvchanlik spektri.

300 nm atrofida yutilish juda yuqori bo'lib, $A \approx 2.0$ – 2.1 ni tashkil etdi (7a-rasm). To'liq uzunligi ortishi bilan absorbsiya keskin kamaydi va 330–350 nm oralig'ida $A \approx 0.3$ – 0.4 gacha tushdi. 350–380 nm sohada absorbsiya egri chizig'ida bukilish kuzatildi. Ushbu holat S va N bilan modifikatsiyalash natijasida hosil bo'lgan lokal energetik holatlar, nuqson sathlari yoki yutilish chekkasi yaqinidagi dum holatlar bilan bog'liq bo'lishi mumkin.

380–400 nm oralig'idan boshlab absorbsiya keskin kamaydi va 400 nm dan keyin $A \approx 0.08$ – 0.12 oralig'ida deyarli barqarorlashdi. Bu ZnO:S:N plyonkasi ko'rinadigan sohada nisbatan shaffofligini saqlab qolganini ko'rsatdi. Biroq sof ZnO bilan solishtirganda absorbsiya fonining

yuqoriroq bo'lishi va optik o'tkazuvchanlikning biroz kamayishi modifikatsiyalangan plyonkada lokal nuqson holatlari ko'payganidan dalolat beradi.

300 nm yaqinida optik o'tkazuvchanlik juda past bo'lib, $T \approx 0$ – 5% atrofida boshlandi (7b-rasm). 300–350 nm oralig'ida transmittans tez ortib, 45–55% ga yetdi. 360–400 nm sohada esa o'tkazuvchanlikning yanada keskin oshishi kuzatildi va 400 nm atrofida 75–82% qiymatlarga yetdi. 400–600 nm oralig'ida transmittans asosan 78–83% diapazonda saqlandi. Ushbu natija ZnO:S:N plyonkasi sof ZnO ga qaraganda biroz pastroq shaffoflikka ega bo'lsa-da, ko'rinadigan sohada yetarli darajada optik o'tkazuvchanlikni saqlab qolishini ko'rsatadi.

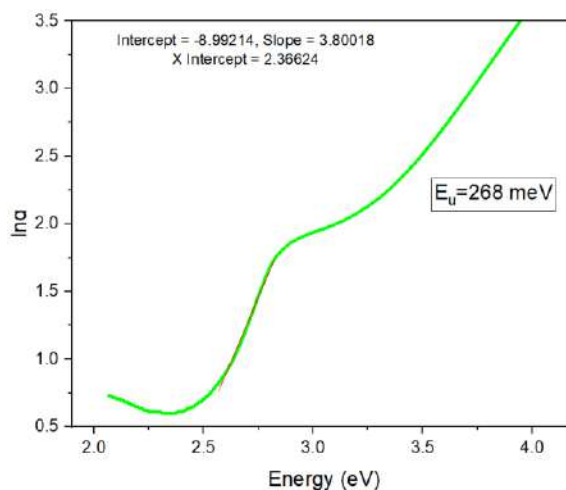


8-rasm. Tauc usuli yordamida ZnO:S:N yupqa plyonkasining optik taqiqlangan zona energiyasini (E_g) aniqlash garafigi.

8-rasmda ZnO:S:N yupqa plyonkasi uchun $(\alpha h\nu)^2$ ning $h\nu$ ga bog'liqligi ko'rsatilgan. Tauc grafigidagi fundamental yutilish chekkasiga mos keluvchi birinchi chiziqli qism absissa o'qigacha ekstrapolyatsiya qilindi. Natijada ZnO:S:N yupqa plyonkasi uchun optik taqiqlangan zona energiyasi $E_g = 3.17$ eV deb aniqlandi. Ushbu qiymat sof ZnO plyonkasidagi $E_g = 3.27$ eV qiymatiga nisbatan kichikroq bo'lib, S va N bilan modifikatsiyalash ZnO ning optik yutilish

chekkasini past energiya tomonga siljitganini ko'rsatadi. Band gap qiymatining kamayishi legirlash bilan bog'liq lokal energetik holatlar, nuqson sathlari va energetik dum holatlarining shakllanishi bilan izohlanishi mumkin[12].

Tauc grafigida yuqori energiyalar sohasida kuzatiladigan qo'shimcha o'sishlar yoki egilishlar plyonkada bir jinsli bo'lmagan optik o'tishlar, sirt/interfeys holatlari yoki legirlovchi atomlar bilan bog'liq lokal elektron holatlar mavjud bo'lishini ko'rsatadi. Shunga qaramay, optik taqiqlangan zona energiyasini aniqlashda fundamental yutilish chekkasiga mos keluvchi birinchi chiziqli qism asosiy mezon sifatida qabul qilindi.



9-rasm. ZnO:S:N yupqa plyonkasi uchun Urbach energiyasini (E_u) aniqlash: $\ln(\alpha)$ ning foton energiyasiga bog'liqligi.

ZnO:S:N yupqa plyonkasida yutilish chekkasi yaqinidagi energetik dum holatlarini baholash uchun Urbach tahlili bajarildi. Buning uchun $\ln(\alpha)$ ning foton energiyasi ($h\nu$) ga bog'liqligi grafigi qurildi. Grafikdagi chiziqli qismning qiyaligi asosida Urbach energiyasi $E_u = 268$ meV deb aniqlandi. Ushbu qiymat sof ZnO

plyonkasidagi $E_u = 74.6$ meV qiymatiga nisbatan ancha katta. Bu esa ZnO:S:N plyonkasida yutilish chekkasi yaqinida sezilarli darajada energetik dum holatlar mavjudligini bildiradi.

Urbach energiyasining ortishi modifikatsiyalangan plyonkada strukturaviy tartibsizlik, donalararo chegaralar, kislorod vakansiyalari, interstitsial nuqsonlar hamda S va N

bilan bog'liq lokal energetik sathlarning kuchayganini ko'rsatishi mumkin. Bunday holat optik yutilish chekkasining kengayishiga va fundamental yutilishning past energiya tomonga siljishiga olib keladi. Shu sababli ZnO:S:N plyonkasida E_g qiymatining 3.17 eV gacha kamayishi va E_u qiymatining 268 meV gacha ortishi o'zaro mos natijalar hisoblanadi.

Umuman olganda, UV-Vis spektroskopiyasi, Tauc va Urbach tahlillari shuni ko'rsatadiki, USP usuli yordamida olingan sof ZnO yupqa plyonkasi yuqori optik shaffoflik va nisbatan past strukturaviy tartibsizlik bilan tavsiflanadi. ZnO:S:N yupqa plyonkasida esa optik o'tkazuvchanlik biroz kamayadi, band gap torayadi va Urbach energiyasi sezilarli ortadi. Bu natijalar S va N bilan modifikatsiyalash orqali ZnO asosidagi yupqa plyonkalarining optik xususiyatlarini maqsadli boshqarish mumkinligini ko'rsatadi.

XULOSA

Ushbu ishda shisha substratlar ustiga ultratovushli purkash pirolizi (USP) usuli yordamida o'stirilgan sof ZnO va ZnO:S:N yupqa plyonkalarining optik xususiyatlari UV-Vis spektroskopiyasi orqali o'rganildi. Olingan natijalar substrat va plyonka tizimlarining optik javoblarining sezilarli farqlanishini va legirlashning ZnO plyonkaning energetik holatiga ta'sirini tasdiqladi. Shisha substrat ko'rinadigan sohada yuqori shaffoflikka ega bo'lib, 400–550 nm oralig'ida 91–92 % optik o'tkazuvchanlikni namoyon etadi, Tauc usuli yordamida optik taqiqlangan zona energiyasi $E_g \approx 3.85$ eV deb topildi.

Sof ZnO plyonkalarda UV sohada kuchli yutilish, ko'rinadigan sohada esa yuqori o'tkazuvchanlik kuzatildi. 370–390 nm oralig'ida yutilish chekkasi mavjud bo'lib, $E_g = 3.27$ eV, Urbach energiyasi esa $E_u = 74.6$ meV bo'lib, plyonkaning yaxshi kristall tuzilishga ega ekanini va ma'lum darajada nuqsonlar mavjudligini ko'rsatdi. ZnO:S:N plyonkalarda optik xususiyatlar sezilarli o'zgardi, UV sohada yanada kuchli yutilish va ko'rinadigan sohada sof ZnO ga nisbatan yuqoriroq absorbsiya foni saqlanib qoldi. Tauc usuli bilan $E_g = 3.17$ eV, Urbach energiyasi $E_u = 268$ meV bo'lib, legirlash natijasida band gap torayganini va optik tartibsizlik ortganini ko'rsatdi.

Umuman olganda, bu ishda USP usuli yordamida yuqori sifatli ZnO va ZnO:S:N plyonkalarini o'stirish mumkinligi isbotlandi. Sof ZnO plyonkalari yuqori shaffoflik va past Urbach energiyasiga ega, ZnO:S:N plyonkalari esa legirlash tufayli band gap torayishi va optik tartibsizlik ortishi bilan ajralib turadi. Natijalar, ZnO asosidagi

plyonkalarni maqsadli ravishda boshqarish va ularni amaliy qurilmalarda qo'llash uchun ilmiy asos bo'lib xizmat qiladi.

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Mualliflar manfaatlar to'qnashuvi mavjud emasligini bildiradilar.

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ADABIYOTLAR

1. S. Nandi, S. Kumar and A. Misra "Zinc oxide heterostructures: advances in devices from self-powered photodetectors to self-charging supercapacitors," (Review Article) Mater. Adv. 2, 6768-6799 (2021) DOI: 10.1039/D1MA00670C
2. Vitalii Karpyna at. all. "Zinc oxide – analogue of GaN with new perspective possibilities," Cryst. Res. Technol. **39**(11) 980 – 992 (2004) DOI:10.1002/crat.200310283
3. P. Ščajev and D. Gogova "Long-lived excitons in thermally annealed hydrothermal ZnO," Heliyon 10(4) e26049 (2024) doi: 10.1016/j.heliyon.2024.e26049
4. Yuldashev Sh.U, Khabibullaev P.K, Nusretov R.A, Khvan I.V, Yalishev V. Sh, Suleymanov S. G. and Kang T.W "Electroluminescence of ZnO/Zn1-xMgxO heterostructures grown on Si substrates," UFJ **9**(6.) 337-341 (2007)
5. Yuldashev Sh. U, Nusretov R. A, Khvan I. V, Yalishev V. Sh, and Kang T. W. "White light emission from ZnO/Zn0.9Mg0.1O heterostructures grown on Si substrates," Japan. Appl. Phys. – **47**(17) 133-135 (2008)
6. S. Zaynabidinov, Sh. Yuldashev, A. Boboev, and N. Yunusaliyev, "X-ray diffraction and electron microscopic studies of the ZnO(S) metal oxide films obtained by the ultrasonic spray pyrolysis method," Herald of the Bauman Moscow State Technical University, Series Natural Sciences, **1**(112), 78-92 (2024). <https://doi.org/10.18698/1812-3368-2024-1-78-92>
7. Bian J.M. Li X.M, Zhang C.Y, Yu W.D, and Gao X.D. "p-type ZnO films by monodoping of nitrogen and ZnO-based p-n homojunctions," Appl. Phys. Lett, **85** 4070 (2004)
8. Y. Ryu at.all "Synthesis of p-type ZnO films," Journal of Crystal Growth **216**(1) 330-334 (200) DOI:10.1016/S0022-0248(00)00437-1
9. K. Salima and W. Azzaoui "Physical properties of spray pyrolysed ZnO thin films obtained from nitrate, acetate and chloride precursors: Comparative study for

- Solar Cell Applications,” *Revista Mexicana de Física* 69 031002 1–8 (2023)
DOI: <https://doi.org/10.31349/RevMexFis.69.031002>
10. R. Kumar, M. Sekhar, Raghvendra, R. Laha, and S. Pandey, “Comparative studies of ZnO thin films grown by electron beam evaporation, pulsed laser and RF sputtering technique for optoelectronics applications,” *Applied Physics*, **126**, 859 (2020).
<https://doi.org/10.1007/s00339-020-04046-8>
11. Zaynabidinov S.Z, Boboev A.Y, Yunusaliyev N.Y. “Effect of γ -irradiation on structure and electrophysical properties of S-doped ZnO films,” *East European Journal of Physics*, **2**, 321-326 (2024).
<https://doi.org/10.26565/2312-4334-2024-2-37>.
12. E. Ahmed, M. Samy and L. Saad “Highly Conductive N-type Aluminum Doped Zinc Oxide for CZTS Kieserite Solar Cell,” *Egypt. J. Chem.* **67**(4) 309 - 313 (2024)
<https://doi.org/10.21608/ejchem.2023.241700.8724>

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KREMNIYIDA KISLORODLI NANOBIRIKMALAR VA LOKAL TUZILMAVIY O'ZGARISHLARNING TAHLILI

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Annotatsiya. Choxralskiy usulida o'stirilgan n-turdagi monokristall kremniyda kislorodli nanobirikmalar va kristall panjaraning lokal buzilishlari o'rtasidagi bog'liqlik tadqiq etildi. Namunalar rentgen diffraksiyasi, to'rt zondli elektr o'lchovlari hamda chuqur sathlarning nostatsionar sig'im spektroskopiyasi (DLTS) yordamida o'rganildi. Tadqiqot uchun (111) yo'nalishdagi, solishtirma qarshiligi 3–10 $\Omega \cdot \text{cm}$ va kislorod konsentratsiyasi $(2-4) \times 10^{17} \text{ cm}^{-3}$ bo'lgan kremniy kristallari tanlandi. Rentgenogrammalarda yuqori intensivlikdagi tor asosiy akslanishlar bilan bir qatorda taqiqlangan (222)Si akslanishi va nomonoton diffuz fon qayd etildi. Ushbu xususiyatlar kislorodning notekis taqsimlanishi hamda panjaradagi lokal mikrodeformatsiyalar bilan bog'liq holda talqin qilindi. SiO_2 fazasining (110) akslanishi asosida kristall nanofragmentlarning xarakterli o'lchami taxminan 80 nm deb baholandi. $2\theta \approx 17,6^\circ$ atrofidagi keng diffuz maksimum esa o'lchami taxminan 1,25 nm bo'lgan amorf SiO_y klasterlariga tegishli deb talqin qilindi. 77–300 K harorat oralig'ida DLTS cho'qqilari kuzatilmadi. Olingan natijalar kremniy matritsasining yuqori kristall mukammalligi saqlangan holda kislorodli nanobirikmalar va lokal tuzilmaviy buzilishlar mavjud bo'lishini ko'rsatadi hamda kislorod tutuvchi kremniyning tuzilmaviy holatini tavsiflash uchun asos yaratadi.

Kalit so'zlar: monokristall kremniy, Choxralskiy usuli, kislorod pretsipitatlari, rentgen diffraksiyasi, SiO_2 nanokristallari, SiO_y klasterlari, tuzilmaviy nuqsonlar, termodonorlar.

Аннотация. Исследована взаимосвязь между кислородсодержащими наносоединениями и локальными искажениями кристаллической решётки в монокристаллическом кремнии n-типа, выращенном методом Чохральского. Образцы изучены методами рентгеновской дифракции, четырёхзондовых электрических измерений и нестационарной ёмкостной спектроскопии глубоких уровней (DLTS). Для исследования использованы кристаллы кремния с ориентацией (111), удельным сопротивлением 3–10 Ом·см и концентрацией кислорода $(2-4) \times 10^{17} \text{ см}^{-3}$. На рентгенограммах наряду с узкими интенсивными основными отражениями зарегистрированы запрещённое отражение (222)Si и немонотонный диффузный фон. Эти особенности интерпретированы как связанные с неоднородным распределением кислорода и локальными микродеформациями решётки. По отражению (110) фазы SiO_2 характерный размер кристаллических нанофрагментов оценён приблизительно в 80 нм. Широкий диффузный максимум при $2\theta \approx 17,6^\circ$ отнесён к аморфным кластерам SiO_y размером около 1,25 нм. В температурном диапазоне 77–300 К пики DLTS не наблюдались. Полученные результаты указывают на существование кислородсодержащих наносоединений и локальных структурных искажений при сохранении высокого кристаллического совершенства кремниевой матрицы и создают основу для характеристики структурного состояния кислородсодержащего кремния.

Ключевые слова: монокристаллический кремний, метод Чохральского, кислородные преципитаты, рентгеновская дифракция, нанокристаллы SiO_2 , кластеры SiO_y , структурные дефекты, термодоноры.

Abstract. The relationship between oxygen-containing nanocompounds and local crystal lattice distortions was investigated in Czochralski-grown n-type single-crystal silicon. The samples were studied using X-ray diffraction, four-point probe electrical measurements, and deep-level transient spectroscopy (DLTS). Silicon crystals with a (111) orientation, a resistivity of 3–10 $\Omega \cdot \text{cm}$, and an oxygen concentration of $(2-4) \times 10^{17} \text{ cm}^{-3}$ were selected for the study. Alongside narrow, high-intensity principal reflections, the X-ray diffraction patterns revealed the forbidden (222)Si reflection and a nonmonotonic diffuse background. These features were interpreted as being associated with a nonuniform oxygen distribution and local lattice microstrain. Based on the (110) reflection of the SiO_2 phase, the characteristic size of the crystalline nanofragments was estimated to be approximately 80 nm. The broad diffuse maximum at $2\theta \approx 17,6^\circ$ was attributed to amorphous SiO_y clusters approximately 1.25 nm in size. No DLTS peaks were observed over the temperature range of 77–300 K. The results indicate that oxygen-containing nanocompounds and local structural distortions coexist with a high degree of crystalline perfection in the silicon matrix, providing a basis for characterizing the structural state of oxygen-containing silicon.

Keywords: single-crystal silicon, Czochralski method, oxygen precipitates, X-ray diffraction, SiO_2 nanocrystals, SiO_y clusters, structural defects, thermodonors.

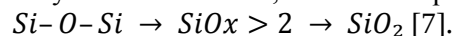
KIRISH

Xomashyoning cheksiz tabiiy zaxiralari, tijoriy jihatdan mavjudligi, o'stirish va plastinalarga

mexanik ishlov berishning qulayligi hamda o'ziga xos elektr xossalari tufayli monokristall kremniy asbobsozlik, elektronika va, ayniqsa, nanoelektronikada asosiy va yetakchi

yarimoʻtkazgich material boʻlib qolmoqda. Texnologik ahamiyati bilan bir qatorda, kremniy kristall materiallarga kirishmalarning kiritilishi, nuqsonlarning hosil boʻlishi va tuzilmaviy oʻzgarishlar bilan bogʻliq fizik-kimyoviy jarayonlarni tadqiq etish uchun model tizim hisoblanadi. Kirishmalar bilan bogʻliq komplekslarning tuzilmaviy holatini tushunish kristall tuzilma bilan yarimoʻtkazgich materiallarning elektr va funksional xossalari oʻrtasidagi bogʻliqliklarni aniqlash uchun muhimdir. Maʼlumki, kremniy monokristallarini sanoat miqyosida oʻstirishning asosiy usuli Choxralskiy usulidir [1, 2]. Hozirgi vaqtda jahonda isteʼmol qilinadigan monokristall kremniyning 80% dan ortigʻi ushbu usul bilan olinadi. Choxralskiy usulining bunday keng tarqalishi uning soddaligi va oʻstirilgan monokristallarning nisbatan yuqori kristall mukammalligi bilan bogʻliq. Choxralskiy usulida kremniy kristallarini oʻstirishning asosiy kamchiliklari kristallarda fon kirishmalari, shuningdek, quymani tortib olishda foydalaniladigan kvars tigel tarkibidagi boshqa kirishmalar miqdorining yuqori boʻlishidir [3]. Fon kirishmalariga kislorod, uglerod va oʻtish metallari — temir, oltin, mis va boshqalar kiradi. Legirlovchi kirishmalarga esa fosfor, bor, mishyak, surma, alyuminiy, kalsiy, rux va boshqalar kiradi; ularning konsentratsiyasi va taqsimlanishi monokristall kremniyning sifatini belgilaydi [4,5]. Fon kirishmalari orasida kremniydagi kislorod muammosi oʻstirish texnologiyasining butun tarixi davomida, yaʼni 60 yildan ortiq vaqt mobaynida dolzarb boʻlib kelmoqda. Kislorod Choxralskiy usulida ($2 \times 10^{17} - 2 \times 10^{18} \text{ cm}^{-3}$) ham, zonali eritish usulida ($2 \times 10^{15} - 2 \times 10^{16} \text{ cm}^{-3}$) ham oʻstirilgan kremniyda asosiy fon kirishmasi hisoblanadi. Uning xatti-harakati katta qiziqish uygʻotadi, chunki u termodonorlarning hosil boʻlishini, zaryad tashuvchilar yashash vaqtining termik barqarorligini, mikronuqsonlarning shakllanishini va getterlash samaradorligini belgilaydi hamda turli fizik va texnologik effektlarning katalizatori sifatida namoyon boʻladi [6]. Fizik kimyo nuqtayi nazaridan kislorod alohida qiziqish uygʻotadi, chunki u nuqsonlar kimyosi, panjaraning lokal buzilishlari, faza hosil boʻlishi va kremniy kristallarining fizik-kimyoviy barqarorligiga taʼsir qiluvchi termik faollashadigan tuzilmaviy oʻzgarishlarda ishtirok etadi. Adabiyotlarda kremniydagi kislorodning xatti-harakatini belgilovchi mexanizmlar boʻyicha yagona fikr mavjud emas. Odatda kislorod kremniyda elektr jihatdan nafaol tugunlararo holatlarni egallaydi va Si-O-Si koʻrinishidagi «bogʻlangan tugunlararo» holatda boʻladi. Kislorodning atomlararo Si-O-Si shaklidan boshqa

elektr jihatdan nafaol shaklga, yaʼni SiO_2 pretsipitatiga oʻtishi elektr jihatdan faol $\text{SiO}_{x>2}$ oraliq holatlarining hosil boʻlishi orqali quyidagi sxema boʻyicha sodir boʻladi, deb taxmin qilinadi:



Choxralskiy usulida oʻstirilgan kristallar kislorod termodonorlari hosil boʻlish tezligi eng yuqori boʻlgan harorat oraligʻida ($< 950 \text{ K}$) termik tavlenganda, ularda zaryad tashuvchilarning yashash vaqti ortishi aniqlangan. Agar kristallarga kislorod termodonorlari parchalanadigan yuqoriroq haroratlarda ($> 950 \text{ K}$) termik ishlov berilsa, zaryad tashuvchilarning yashash vaqti odatda kamayadi [8]. Shunday qilib, kislorod oʻzining konsentratsiyasi, taqsimlanishi va mavjudlik shakliga qarab kremniy monokristallarining tuzilmaviy hamda elektrofizik xossalariga ijobiy ham, salbiy ham taʼsir koʻrsatishi mumkin [9].

Hozirgi vaqtda monokristall kremniydagi kislorodning fazaviy holati va mavjudlik shakllari skanerlovchi elektron va atom-kuch mikroskopiyasi, selektiv yemirish, infraqizil spektroskopiya hamda yuqori ajratuvchanlikdagi rentgen difraksiyasi kabi turli usullar bilan oʻrganilmoqda. Ushbu usullar orasida rentgen difraksiyasi yarimoʻtkazgich materiallarning kristall tuzilmasi, fazaviy tarkibi, panjara deformatsiyasi va nuqsonlar sababli yuzaga keladigan tuzilmaviy oʻzgarishlarini tadqiq etishda eng koʻp maʼlumot beruvchi usullardan biri boʻlib qolmoqda. U kirishmalar taqsimoti, tuzilmaviy mukammallik va kristall panjara ichida sodir boʻladigan fizik-kimyoviy oʻzgarishlar oʻrtasidagi bogʻliqlik haqida muhim maʼlumot beradi. Quyida Choxralskiy usulida oʻstirilgan monokristall n-Si ning tuzilmasini oʻrganish boʻyicha eksperimental natijalar keltirilgan. Tadqiqotning maqsadi takomillashtirilgan optik tizimga ega rentgen difraktometri yordamida monokristallning tuzilmaviy mukammalligini va kremniy panjarasidagi kislorodning mavjudlik shaklini aniqlashdan iborat. Kislorod bilan bogʻliq tuzilmaviy nuqsonlarni aniqlash, panjara buzilishlarini baholash hamda kremniy matritsasida hosil boʻlgan kislorod tutuvchi fazalar va komplekslarning fizik-kimyoviy tabiatini aniqlashtirishga alohida eʼtibor qaratildi.

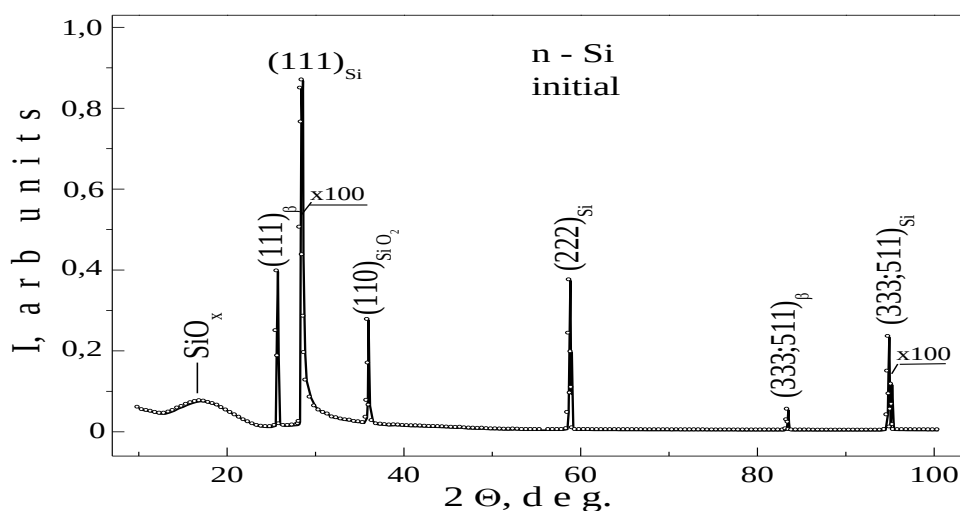
MATERIALLAR VA METODLAR

Tadqiqot obyekti sifatida Choxralskiy usulida oʻstirilgan, solishtirma qarshiligi $\rho \approx (3-10) \Omega \cdot \text{cm}$, dislokatsiyalar zichligi $N_D \approx 10^1 - 10^3 \text{ cm}^{-2}$ va kislorod konsentratsiyasi $N_O \approx 2-4 \times 10^{17} \text{ cm}^{-3}$ boʻlgan n-Si (CEF) monokristallari tanlandi. Oʻlchashlardan oldin namunalarning taxminan $20 \mu\text{m}$ qalinlikdagi sirt qatlamini olib tashlash uchun ularga mexanik va kimyoviy sayqal

berildi. Namunalarning ishchi sirti (111) yoʻnalishga ega boʻlib, uning oʻlchamlari $5 \times 15.8 \text{ mm}^2$, namuna qalinligi esa 0.7 mm ni tashkil etdi. Rentgen-difraksiya tadqiqotlar 6100 turidagi rentgen difraktometrida (XRD-6100, Shimadzu, Yaponiya, $\text{CuK}\alpha$ nurlanishi, $\lambda = 0.15418 \text{ nm}$) $10\text{--}100^\circ$ skanerlash oraligʻida, $2^\circ/\text{min}$ tezlikda bajarildi. Tushuvchi nur intensivligi taxminan 5×10^5 impuls/s, intensivlikni oʻlchash xatoligi esa $\sim 0.6\%$ ni tashkil etdi. Solishtirma qarshilik standart toʻrt zondli usulda xona haroratida oʻlchandi. Noasosiy zaryad tashuvchilar konsentratsiyasi chuqur sathlarning nostatsionar sigʻim spektroskopiyasi (DLTS) usuli yordamida aniqlandi [10, 11].

NATIJALAR MUHOKAMASI

1-rasmda dastlabki monokristall n-Si ning rentgenogrammasi keltirilgan. Rentgenogrammada

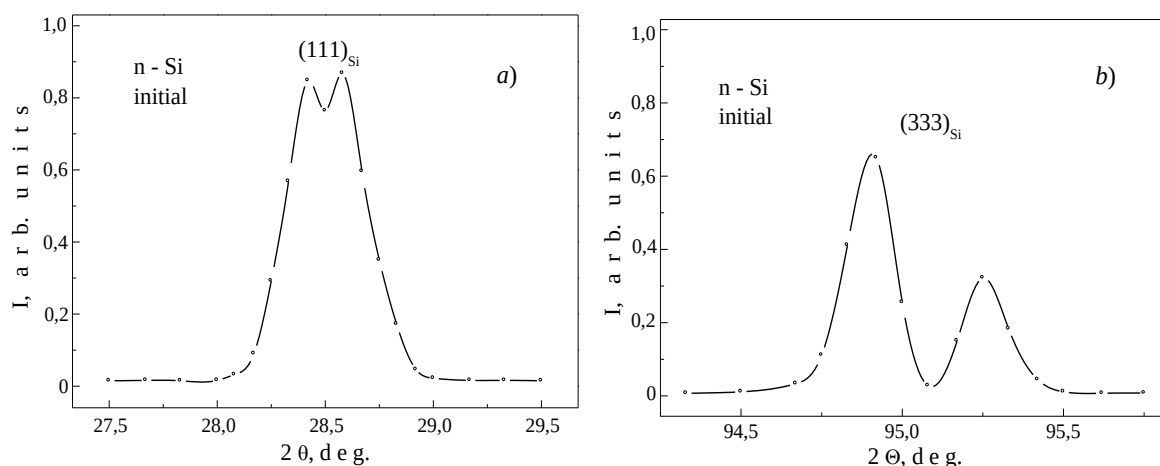


1-rasm. Dastlabki monokristall n-Si ning rentgenogrammasi.

(111)_{Si} difraksiya choʻqqisining yuqori intensivligi (2×10^5 impuls/s) va tor kengligi ($FWHM = 2.62 \times 10^{-3} \text{ rad}$) dastlabki monokristall panjarasining yuqori mukammalligini koʻrsatadi (2a-rasm). Biroq ushbu akslanishning deyarli singlet koʻrinishi - rentgenogrammada faqat α_1 va α_2 nurlarga ajralish izlari kuzatilishi - matritsa panjarasida oʻsish jarayonida yuzaga kelgan lokal elastik mikrokuchlanishlar mavjudligini bildiradi. Bu kislorodning kremniy panjarasi tugunlari orasida notekis taqsimlanishi bilan bogʻliq [12]. Mazkur xulosa rentgenogrammada kremniy panjarasi uchun taqiqlangan, $d/n = 0.1568 \text{ nm}$ boʻlgan (222)_{Si} akslanishining mavjudligi bilan tasdiqlanadi. Uning intensivligining asosiy (111)_{Si} choʻqqisi intensivligiga nisbati $I(222)/I(111) = 1.94 \times 10^{-2}$ ni tashkil etadi. Bu kislorodning matritsa panjarasida bir tekis taqsimlanishiga mos keladigan 10^{-4}

turli intensivlikdagi bir nechta selektiv tuzilmaviy choʻqqilar, shuningdek, kichik sochilish burchaklarida diffuz sochilish mavjudligi koʻrinadi. Tahlil monokristall sirti (111) kristallografik tekislikka mos kelishini koʻrsatdi. Buni rentgenogrammada (HHH) turidagi (bu yerda H = 1, 2, 3...) selektiv akslanishlar qatori mavjudligi tasdiqlaydi: $d/n = 0.3121$ ($2\theta = 28.7^\circ$) boʻlgan (111)_{Si}, $d/n = 0.1568$ ($2\theta = 58.9^\circ$) boʻlgan (222)_{Si} va $d/n = 0.1046 \text{ nm}$ ($2\theta = 94.9^\circ$) boʻlgan (333)_{Si} intensiv chiziqlari. Birinchi tartibli (111)_{Si} tuzilmaviy chizigʻining beta (β) komponenti $2\theta = 25.8^\circ$ sochilish burchagida, uchinchi tartibli (333)_{Si} chizigʻining β komponenti esa $2\theta = 83.7^\circ$ burchakda kuzatiladi.

qiymatdan ancha katta [13]. Ushbu dalillar kislorodning kremniy panjarasi tugunlari orasida notekis taqsimlanganini va buning natijasida panjarada mikrodeformatsiyalarga ega lokal sohalar mavjudligini koʻrsatadi. Difraksiya soʻnish qonuniga koʻra, buzilmagan olmos tipidagi kremniy panjarasining rentgenogrammasida (222)_{Si} akslanishi kuzatilmagligi kerak. U faqat panjarada buzilishlar, yaʼni elastik mikrokuchlanishlar mavjud boʻlganda paydo boʻladi. (333)_{Si} akslanishining hisoblangan qiymatga ($3.79 \times 10^{-3} \text{ rad}$) yaqin darajada yaxshi ajralishi hamda α_1 va α_2 nurlanishlar uchun xos intensivliklar nisbatining $[I_{333}(\alpha_1) \approx 2 \times I_{333}(\alpha_2)]$ kuzatilishi (2b-rasm) elastik kuchlanish asosan monokristall kremniyning sirtga yaqin qatlamida, dislokatsiyalarda hamda bloklar (subkristallitlar) chegaralarida toʻplanganini koʻrsatadi.



2-rasm. Monokristall n-Si ning (111) (a) va (333) (b) akslanishlari profilari.

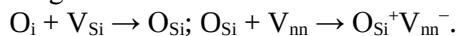
Bu xulosa rentgenogrammada kremniydagi SiO_2 kirishma fazasi bilan bog'liq, $d/n = 0.2495$ nm ($2\theta = 36.0^\circ$) bo'lgan (110) akslanishining mavjudligi bilan tasdiqlanadi. Ushbu akslanishning tor kengligi ($FWHM = 1.89 \times 10^{-3} rad$) kremniy dioksidi — SiO_2 fazasining kristall holatda ekanligini ko'rsatadi. Kirishma fazaning tuzilmaviy fragmentlari — nanokristallarining xarakterli o'lchami ≈ 80 nm ni tashkil etdi. Ushbu akslanish intensivligining matritsa panjarasi asosiy $(111)_{Si}$ akslanishi intensivligiga nisbati $I(110)_{SiO_2}/I(111)_{Si} = 2.47 \times 10^{-3}$ bo'lib, bu kislorodning kremniy panjarasida bir jinsli taqsimlanishiga mos keluvchi 10^{-4} qiymatdan katta. Bu holat kislorodning notekis taqsimlanganini va monokristall panjarasida kremniy ($r = 0.040$ nm) hamda kislorod ($r = 0.140$ nm) ion radiuslarining farqi bilan bog'liq elastik kuchlanishlar mavjudligini yana bir bor tasdiqlaydi [14, 15]. Kislorod ion radiusining kremniylikiga nisbatan katta bo'lishi tufayli kislorod ionlarining monokristall donalari va bloklari orasidagi chegaraviy hamda donalararo sohalarining tugunlararo holatlarida, ya'ni matritsa panjarasining nuqsonlarga moyil sohalarida joylashishi energetik jihatdan qulayroqdir. Donalar (bloklar) orasidagi qatlamlarda yuqori potentsialli tugunlar mavjud bo'lgani sababli, ushbu sohalaridagi matritsa ionlari orasidagi kimyoviy bog'lanishlar to'yinmagan bo'lishi, ayrim tugunlar esa kubik simmetriyaga mos ideal holatlariga nisbatan biroz siljigan holatda bo'lishi mumkin. Ushbu omillar monokristall bloklari va donalari hajmida faqat kremniy ionlaridan tashkil topgan yuqori (kubik) simmetriyali tuzilmaga nisbatan, nuqsonlarga moyil sohalarida kremniy hamda tugunlararo, mazkur holatda kislorod ionlaridan iborat past simmetriyali tuzilmaviy fragmentlarning hosil bo'lishiga yordam beradi.

Bundan tashqari, rentgenogrammada kichik sochilish burchaklarida ($2\theta \approx 17.6^\circ$) keng

($FWHM = 116.36 \times 10^{-3} rad$) diffuz akslanish kuzatiladi. Ushbu akslanish sirtga yaqin qatlamlardagi to'yinmagan bog'lanishlarga ega SiO_x tuzilmaviy fragmentlariga tegishli deb qaraladi. Akslanishning juda keng bo'lishi uni yuzaga keltiruvchi tuzilmaviy fragmentlar o'lchamining kichikligini va ularning joylashuvida uzoq tartib mavjud emasligini ko'rsatadi. Demak, mazkur tuzilmaviy fragmentlar SiO_x nanokristallari emas, balki klasterlardir. Ushbu klasterlarning xarakterli o'lchami ~ 1.25 nm ($12.5 \text{ \AA}$) ni tashkil etdi. Monokristallning eksperimental aniqlangan panjara parametri $a_s = 0.54331$ nm bo'lib, $a_{Si} = 0.54307$ nm jadval qiymatiga yaqin. Bu buzilgan panjara sohasi hajmining ulushi monokristallning umumiy hajmiga nisbatan juda kichik ekanligini ko'rsatadi.

Tadqiq etilgan namunalarning eksperimental o'lchangan solishtirma qarshilik qiymatlari $7.28\text{--}7.60 \text{ } \Omega \cdot \text{cm}$ oralig'ida bo'ldi. Dinamik sig'imning $77\text{--}300$ K harorat oralig'idagi haroratga bog'liqligi abssissa o'qiga parallel gorizontaal chiziq ko'rinishida namoyon bo'ldi. Ya'ni DLTS spektrlarida tadqiq etilgan n-Si namunalarda p-n o'tish mavjudligini ko'rsatuvchi xarakterli maksimum kuzatilmadi. Ushbu holatning ehtimoliy izohlaridan biri kislorodning kremniy panjarasidagi joylashuvining o'zgarishidir. Ma'lumki, tugunlararo Si-O-Si holatida kislorod kremniy ionlari bilan to'yingan bog'lanishlar hosil qiladi va shu sababli elektr jihatdan nafaol bo'ladi [16, 17]. Elektr jihatdan faol bo'lishi, ya'ni termodonor vazifasini bajarishi uchun kislorod ionlari kremniy panjarasi tugunlarini egallashi, boshqacha aytganda, kremniy ionlarining o'rnini bosishi kerak. 450°C da past haroratli termik tavlash vaqtida tugunlararo kislorod konsentratsiyasining kamayishi va termodonorlar konsentratsiyasining ortishi, ya'ni to'yinmagan bog'lanishlarga ega kislorod va kremniy ionlari komplekslari — SiO_y ($y > 2$) hosil bo'lishi kuzatilgani ko'rsatilgan. Umuman olganda, kislorodning tugunlararo

shakldan pretsipitat shakliga (SiO_2) o'tishi bir nechta oraliq holatlarning hosil bo'lishi orqali sodir bo'lishi mumkin: $\text{Si}-\text{O}-\text{Si} \rightarrow \text{SiO}_{y>2} \rightarrow \text{SiO}_2$. Ushbu oraliq holatlarning ayrimlari elektr jihatdan faol bo'lishi, ya'ni kristallda donor holatlarini yuzaga keltirishi mumkin, deb taxmin qilinadi; bunda termodonor turidagi tuzilmaviy fragmentlar hosil bo'ladi. Kremniy monokristallarini o'stirish vaqtida termodonorlar hosil bo'lish ehtimoli juda yuqori ekanini ta'kidlash kerak, chunki sekin soviyotgan quyma 300–500 °C harorat oralig'idan o'tadi. Binobarin, $\text{SiO}_{y>2}$ oraliq holatlarining va SiO_2 pretsipitatlarining aksariyati elektr jihatdan nofaol bo'lib, elektr jihatdan faol bo'lganlari esa kristallda donor holatlarini hosil qiladi [18]. Shuning uchun ushbu tuzilmaviy fragmentlar elektr jihatdan nofaol yoki faol holatda bo'lishidan qat'i nazar, n-Si monokristallari hajmida mavjud bo'lganda p–n o'tish hosil bo'lishiga hissa qo'shmaydi. Bu tajribalarda kuzatilganidek, ushbu monokristallarning DLTS spektrlarida xarakterli cho'qqi mavjud emasligini izohlaydi. [19] ga ko'ra, n-Si monokristallarining DLTS spektrida xarakterli cho'qqi kuzatilmassligining yana bir ehtimoliy sababi kislorod va vakansiyalarning bevosita o'zaro ta'siri natijasida quyidagi reaksiyalar orqali donorlarning hosil bo'lishidir:



Bu yerda O_i — tugunlararo kislorod; O_{Si} — kremniy panjarasi tugunidagi kislorod ioni; V_{nn} — tugundagi kislorod yonida, ya'ni qo'shni tugunda joylashgan vakansiya. Birinchi reaksiya tugunlararo kislorodni tugundagi kislorodga aylantiradi. Ikkinchi reaksiyada ushbu tugundagi kislorod ioni eng yaqin qo'shni vakansiya (V_{nn}) bilan birikadi. Bunda kislorod o'z valent elektronlaridan birini vakansiyaga «beradi» va quyidagi «reaksiya» bo'yicha termodonor vazifasini bajarishi mumkin:



Biroq mavjud tuzilmaviy usullar yordamida termodonorlar hosil bo'lishining ikkinchi mexanizmi to'g'riligini tasdiqlash qiyinchilik tug'diradi.

XULOSA

Shunday qilib, olingan eksperimental natijalarni quyidagicha umumlashtirish mumkin:

– Tadqiq etilgan kremniy monokristallari mukammal tuzilishga ega. Buni {HHH} turidagi tuzilmaviy akslanishlarning tor kengligi ($\approx 10^{-3} \text{ rad}$) va yuqori intensivligi ($\approx 10^4 - 10^5 \text{ imp} \cdot \text{s}^{-1}$) tasdiqlaydi.

– Kichik va o'rta sochilish burchaklarida ($2\theta < 60^\circ$) noelastik fonning nomonoton xususiyati kislorodning kremniy matritsasi panjarasida notekis taqsimlanganini ko'rsatadi.

– Monokristall o'sishi davomida uning nuqsonlarga moyil sohalarida, ya'ni donalar orasidagi chegaraviy qatlamlarda SiO_2 (kremniy dioksidi) stexiometrik tarkibli kislorod pretsipitatlari hosil bo'lishi mumkinligi ko'rsatildi. Buni difraksion spektrda ushbu fazaga tegishli (110) akslanishining mavjudligi tasdiqlaydi. SiO_2 pretsipitatlari rombik simmetriyaga ega bo'lib, o'lchami $\approx 80 \text{ nm}$ bo'lgan nanokristallardir. Buni (110) tuzilmaviy akslanishining tor kengligi ($1.89 \times 10^{-3} \text{ rad}$) ko'rsatadi.

– Monokristall o'sishi davomida donor xossalarga ega $\text{SiO}_{y>2}$ ko'rinishidagi kislorod–kremniy komplekslarining oraliq holatlari hosil bo'lishi mumkinligi haqida taxmin ilgari surildi. Namunalarda ularning mavjudligi difraksion spektrdagi SiO_2 kirishma fazasiga tegishli tuzilmaviy akslanish orqali namoyon bo'ladi, chunki SiO_2 pretsipitatlari ushbu komplekslarning yakuniy mahsulotlaridir.

– Kremniy monokristalli sirtida tuzilmaviy fragmentlari 1.25 nm o'lchamga ega amorf oksid qatlami aniqlandi. Ushbu qatlam difraksion spektrning kichik sochilish burchaklarida keng ($116.36 \times 10^{-3} \text{ rad}$) diffuz akslanishni yuzaga keltiradi.

ADABIYOTLAR

1. H. J. Jeon, H. Park, et al., Processes 8, 1077 (2020). <https://doi.org/10.3390/pr8091077>
2. A. Y. Boboev, B. M. Ergashev, N. Y. Yunusaliyev, and M. M. Xotamov, East Eur. J. Phys. (2), 292–296 (2025). <https://doi.org/10.26565/2312-4334-2025-2-36>
3. A. R. Ansari Dezfoli and Z. Adabavazeh, Heliyon 10, e29346 (2024). <https://doi.org/10.1016/j.heliyon.2024.e29346>
4. K. Kakimoto, X. Liu, and S. Nakano, Materials 15, 1843 (2022). <https://doi.org/10.3390/ma15051843>
5. S. Nakano, X. Liu, X.-F. Han, and K. Kakimoto, Crystals 11, 27 (2021). <https://doi.org/10.3390/cryst11010027>
6. A. Y. Boboev, B. M. Ergashev, N. Y. Yunusaliyev and J. S. Madaminjonov, East European Journal of Physics 2, 431 (2025). <https://doi.org/10.26565/2312-4334-2025-2-53>
7. Q. Zhang, A. Wang, et al., Mater. Sci. Semicond. Process. 192, 109419 (2025). <https://doi.org/10.1016/j.mssp.2025.109419>
8. E. Olsen, T. Mehl, et al., Phys. Status Solidi A 219, 2100655 (2022). <https://doi.org/10.1002/pssa.202100655>
9. D. Wu, T. Zhao, et al., J. Appl. Phys. 138, 125703 (2025). <https://doi.org/10.1063/5.0292095>
10. Z. Wang, X. Zhu, et al., Sol. Energy Mater. Sol. Cells 236, 111533 (2022). <https://doi.org/10.1016/j.solmat.2021.111533>

11. C. Liu, W. Dou, et al., J. Mater. Sci. 58, 10651 (2023).
<https://doi.org/10.1007/s10853-023-08675-1>
12. S. Utamuradova, S. Daliev, J. Khamdamov, K. Matchonov, and A. Khaitbaev, East Eur. J. Phys. (4), 484 (2025). <https://doi.org/10.26565/2312-4334-2025-4-49>
13. I. R. Entin and I. A. Smirnova, Acta Crystallogr. A 45, 577 (1989).
<https://doi.org/10.1107/S0108767389003879>
14. M. U. Kalanov, I. I. Sadikov, et al., Silicon 16, 4091 (2024).
<https://doi.org/10.1007/s12633-024-02985-y>
15. Z. Li, J. Will, and D. Yang, AIP Adv. 10, 105324 (2020).
<https://doi.org/10.1063/5.0027232>
16. F. S. Gentile, A. Difalco, et al., J. Chem. Phys. 152, 054502 (2020).
<https://doi.org/10.1063/1.5139564>
17. Z. Cun, Q. Jin, and S. Li, J. Cryst. Growth 648, 127898 (2024).
<https://doi.org/10.1016/j.jcrysgro.2024.127898>
18. M. Siriwardhana, F. Rougieux, et al., IEEE J. Photovolt. 12, 222 (2022).
<https://doi.org/10.1109/JPHOTOV.2021.3126120>
19. A. Platonenko, F. Colasuonno, et al., J. Chem. Phys. 154, 174707 (2021).
<https://doi.org/10.1063/5.0044106>

Upwind and high-order difference schemes for nonlinear hyperbolic systems with nonlocal characteristic velocities: construction, accuracy and application to a production-line model

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Abstract: Hyperbolic systems whose characteristic speeds depend not on the local value of the solution but on an integral functional of it over the whole domain arise in continuum models of highly re-entrant manufacturing lines, supply chains and traffic flows. Difference schemes for local hyperbolic systems are well developed, whereas the numerical analysis of systems with nonlocal speeds of both signs, coupled through dissipative feedback boundary conditions and nonlinear source terms, has received much less attention. The aim of this paper is to construct and analyse a family of difference schemes for one-dimensional nonlinear hyperbolic systems in Riemann invariants with nonlocal characteristic speeds, which serves as the basis for a subsequent discrete Lyapunov stability analysis. Three schemes are built: an explicit upwind scheme in which the nonlocal speed is frozen on the old time level and the boundary conditions are imposed as a discrete feedback on the new level; a family of second-order TVD schemes based on MUSCL reconstruction with minmod, van Leer and Fromm limiters and two-stage SSP Runge–Kutta time stepping; and an implicit scheme in which the nonlocal speed is taken on the new time level and the resulting nonlinear equation is solved by Picard iteration. We prove that the error of the quadrature rule used for the nonlocal functional turns into a speed error and limits the global order of the scheme to $\min(r, p)$, where r is the order of the scheme and p is the order of the quadrature. For the explicit scheme the CFL condition is stated with respect to the upper bound of the admissible speeds, and monotonicity together with a discrete energy inequality is established. The TVD condition $v \leq 1/2$ is proved for the high-order schemes, and a mesh-independent contraction condition is obtained for the Picard iteration of the implicit scheme.

Keywords: nonlocal velocity, hyperbolic system, upwind scheme, MUSCL, TVD, quadrature rule, Picard iteration, re-entrant production line.

Аннотация: Для нелинейных гиперболических систем, характеристические скорости которых зависят от интегрального функционала решения, построены явная противопоточная схема, TVD-схемы второго порядка (MUSCL, SSP Рунге–Кутта) и неявная схема. Доказано, что погрешность квадратуры ограничивает порядок схемы величиной $\min(r, p)$; получены условие TVD $v \leq 1/2$ и не зависящее от сетки условие сходимости итераций Пикара. Результаты подтверждены численно и применены к модели производственной линии с контуром переработки брака.

Ключевые слова: нелокальная скорость, гиперболическая система, противопоточная схема, MUSCL, TVD, квадратурная формула, итерации Пикара, производственная линия.

Annotatsiya: Maqolada xarakteristik tezliklari yechimning integral funksionaliga bog'liq bo'lgan nochiziqli giperbolik sistemalar uchun oqimga qarshi, MUSCL–TVD va oshkormas ayirmali sxemalar quriladi. Kvadratura xatoligi sxema tartibini $\min(r, p)$ gacha cheklashi isbotlangan, TVD sharti $v \leq 1/2$ va Pikar iteratsiyasining to'rga bog'liq bo'lmagan yaqinlashish sharti olingan. Natijalar aniq yechimli testda tasdiqlangan va qayta ishlov halqasiga ega ishlab chiqarish liniyasi modeliga tatbiq qilingan.

Kalit so'zlar: nolokal tezlik, giperbolik sistema, oqimga qarshi sxema, MUSCL–TVD sxema, kvadratura formulasi, Pikar iteratsiyasi, qayta kiruvchi ishlab chiqarish liniyasi, WIP.

1. Introduction

Hyperbolic systems whose characteristic speeds depend not on the local value of the solution but on an integral functional of the solution over the whole domain have been actively studied in recent years. A classical example is the continuum model of a highly re-entrant production line in semiconductor wafer fabrication, in which the processing speed decreases as the total work in progress (WIP) on the line increases [1–4]. Similar structures arise in models of supply chains, traffic flows and crowd dynamics [5–7].

The theory of difference schemes for local hyperbolic systems is well developed: the Godunov scheme [8], upwind schemes and their high-order extensions [9–12], TVD limiters [13, 14] and strong stability preserving (SSP) Runge–Kutta methods [15]. Finite volume schemes for nonlocal

conservation laws and their high-order variants were constructed in [6, 16, 17]. However, these works deal mainly with scalar equations and nonlocal fluxes (convolutions). In the present paper we study systems with speeds of both signs, in which the speed is determined by a functional taken over the whole domain and the components are coupled through dissipative boundary conditions and nonlinear source terms.

The stability of upwind splitting schemes for symmetric t -hyperbolic systems [18–20] and the exponential stability of systems with nonlocal speeds established by means of discrete Lyapunov functions [21, 22] were studied previously.

The main results of the paper are as follows.

1. The influence of the quadrature error for the nonlocal functional on the order of the scheme is established: the overall order equals $\min(r, p)$ (Proposition 1). This is confirmed by a numerical experiment.
2. Explicit upwind, MUSCL–TVD and implicit schemes approximating dissipative boundary conditions are constructed for a system with nonlocal speeds of both signs. For these schemes, the stability conditions $v \leq 1$ and $v \leq 1/2$, as well as a mesh-independent convergence condition for the Picard iteration, are obtained (Lemmas 2–4).
3. A nonlocal hyperbolic model of a production line with a rework loop is formulated. Its equilibria and operating curve are computed accurately, numerical solutions are obtained for dynamic scenarios, and the schemes are compared with the exact and reference solutions.

2. Problem statement and discretization of the nonlocal operator

2.1. The continuous problem

In the domain $(0, 1) \times (0, \infty)$, we consider the following system written in Riemann invariants:

$$\partial_t u_i + \lambda_i(W(t)) \partial_x u_i = F_i(u), \quad i = 1, \dots, n, \quad (1)$$

$$W(t) = W_0 + \int_0^1 \omega(x) \varphi(u(x, t)) dx. \quad (2)$$

Here $\lambda_i \in C^1$, φ and F are Lipschitz functions, and $\omega \geq 0$ is a weight function. The components with $i \leq m$ move to the right ($\lambda_i > 0$), and those with $i > m$ move to the left ($\lambda_i < 0$). The boundary conditions are given in feedback form:

$$y_{\text{in}}(t) = \mathcal{G}(y_{\text{out}}(t), W(t), t), \quad (3)$$

where y_{in} denotes the incoming values ($u_i(0, t)$ for $i \leq m$ and $u_i(1, t)$ for $i > m$) and y_{out} denotes the outgoing values. The initial condition $u(x, 0) = u^0(x)$ is assumed to be compatible with (3).

Assumptions. (A1) On the admissible set, $0 < \lambda_* \leq |\lambda_i(W)| \leq \lambda^*$ and $|\lambda_i(W) - \lambda_i(\tilde{W})| \leq L_\lambda |W - \tilde{W}|$. (A2) $|F(u) - F(\tilde{u})| \leq L_F |u - \tilde{u}|$. (A3) $\mathcal{G}$ is Lipschitz continuous with respect to its arguments.

2.2. Grid and grid functions

We introduce the spatial grid $\bar{\omega}_h = \{x_j = jh, j = 0, \dots, J\}$, $h = 1/J$, and the time grid $t_n = n\tau$, and denote $u_{i,j}^n \approx u_i(x_j, t_n)$. The discrete norms are defined as follows:

$$\|v\|_h^2 = h \sum_j v_j^2, \quad \|v\|_\infty = \max_j |v_j|.$$

The nonlocal functional (2) is approximated by a quadrature rule with nonnegative weights:

$$W_h^n = W_0 + h \sum_{j=0}^J \theta_j \omega(x_j) \varphi(u_j^n). \quad (4)$$

For the left rectangle rule, $\theta_j = 1$ ($j < J$) and $\theta_j = 0$ ($p = 1$). For the trapezoidal rule, $\theta_0 = \theta_J = \frac{1}{2}$, and the remaining weights are equal to 1 ($p = 2$). For Simpson's rule (J even), $\theta = \frac{1}{3}(1, 4, 2, \dots, 2, 4, 1)$ ($p = 4$).

Lemma 1 (transfer of the quadrature error to the speed). If $\omega\varphi(u(\cdot, t)) \in C^p[0, 1]$ and the quadrature rule is of order p , then on the exact solution

$$|\lambda_i(W_h) - \lambda_i(W)| \leq L_\lambda C_q h^p, \quad C_q = c_p \max_x |\partial_x^p(\omega\varphi(u))|. \quad (5)$$

Proof. Estimate (5) follows directly from the standard bound for the quadrature remainder and the Lipschitz condition in (A1). $\square$

Proposition 1. Suppose that the scheme is accurate of order r when used with the exact W . Then, when W_h is used, the approximation error is $O(h^r + h^p)$. If the scheme is stable, the global error is also $O(h^{\min(r, p)})$.

Proof. Replacing $\lambda(W)$ by $\lambda(W_h)$ in equation (1) adds the extra term $(\lambda(W_h) - \lambda(W)) \partial_x u = O(h^p)$ to the local truncation error. The rest of the proof follows the standard Lax-type argument.

Hence, using the left rectangle rule in a second-order scheme reduces the overall accuracy to first order. This result is confirmed numerically in Section 6 (Fig. 2b).

3. Upwind difference scheme

We freeze the nonlocal speeds on each time level: $\lambda_i^n = \lambda_i(W_h^n)$, $\sigma_i^n = \tau |\lambda_i^n|/h$. The explicit upwind scheme has the form

$$\begin{aligned} u_{i,j}^{n+1} &= (1 - \sigma_i^n) u_{i,j}^n + \sigma_i^n u_{i,j-1}^n + \tau F_i(u_j^n), \quad j = 1, \dots, J, \quad i \leq m, \\ u_{i,j}^{n+1} &= (1 - \sigma_i^n) u_{i,j}^n + \sigma_i^n u_{i,j+1}^n + \tau F_i(u_j^n), \quad j = 0, \dots, J-1, \quad i > m, \\ y_{\text{in}}^{n+1} &= \mathcal{G}(y_{\text{out}}^{n+1}, W_h^{n+1}, t_{n+1}). \end{aligned} \quad (6)$$

The interior nodes are updated using only the data on time level n . Therefore, y_{out}^{n+1} is known before the boundary map is applied, and the scheme is fully explicit. The boundary condition (3) is imposed on the new time level as a discrete feedback.

Courant–Friedrichs–Lewy condition. Since the nonlocal speed varies in time, the CFL condition is imposed with respect to the upper bound of the set of admissible speeds:

$$v := \frac{\tau}{h} \sup_w \max_i |\lambda_i(W)| \leq 1. \quad (7)$$

Lemma 2 (monotonicity). If condition (7) holds and $F \equiv 0$, then every interior step of scheme (6) is a convex combination of neighbouring values. Consequently, the maximum principle $\min_j u_{i,j}^n \leq u_{i,j}^{n+1} \leq \max_j u_{i,j}^n$ holds (at the interior nodes), and for each family the inequality $\sum_j (u_{i,j}^{n+1})^2 \leq (1 - \sigma_i^n) \sum_j (u_{i,j}^n)^2 + \sigma_i^n \sum_j (u_{i,j-1}^n)^2$ is valid. If $F \neq 0$, an additional term $O(\tau L_F)$ appears.

Proof. Since $0 \leq \sigma_i^n \leq 1$, the coefficients are nonnegative and sum to one. The second inequality follows from the convexity of the function $s \mapsto s^2$. $\square$

The inequality of Lemma 2 is the starting point for constructing an exponentially weighted discrete Lyapunov function in the subsequent stability analysis. The approximation order of scheme (6) is $O(\tau + h)$ (for $p \geq 1$, by Proposition 1).

4. High-order schemes

For the right-moving family ($\lambda > 0$), the values at the cell interfaces are recovered from the nodal values by linear reconstruction:

$$u_{j+1/2} = u_j + 1/2 s_j, \quad s_j = \phi(\Delta_- u_j, \Delta_+ u_j), \quad \Delta_\pm u_j = \pm(u_{j\pm 1} - u_j). \quad (8)$$

The limiter functions are chosen as follows:

$$\begin{aligned}\phi_{\text{mm}}(a, b) &= \begin{cases} \text{sign}(a) \min(|a|, |b|), & ab > 0, \\ 0, & ab \leq 0, \end{cases} & \phi_{\text{vL}}(a, b) \\ &= \begin{cases} \frac{2ab}{a+b}, & ab > 0, \\ 0, & ab \leq 0, \end{cases} & \phi_{\text{F}}(a, b) = \frac{a+b}{2}.\end{aligned}\quad (9)$$

They yield the minmod, van Leer and unlimited (Fromm) schemes, respectively. The spatial operator is written as $L_h(u)_j = -\lambda(W_h)(u_{j+1/2} - u_{j-1/2})/h + F(u_j)$. For time integration, the two-stage SSP Runge–Kutta method (Heun's method) is used [15]:

$$u^{(1)} = u^n + \tau L_h(u^n), \quad u^{n+1} = 1/2 u^n + 1/2 (u^{(1)} + \tau L_h(u^{(1)})). \quad (10)$$

At each stage, W_h is recomputed from the solution at that stage and the boundary condition (3) is applied. For the left-moving families, (8) is written with mirrored indices. The ghost nodes outside the domain are defined by linear extrapolation.

Lemma 3 (TVD condition). Let $F \equiv 0$. Then, under the condition

$$v \leq 1/2 \quad (11)$$

the scheme constructed from reconstruction (8)–(9) and the explicit Euler step does not increase the total variation (TVD). By the SSP property, scheme (10) is also TVD under the same condition.

Proof. We write the Euler step in the form $u_j^{n+1} = u_j - C_{j-1/2} \Delta_- u_j$, where

$$C_{j-1/2} = v \left(1 + \frac{1}{2} \frac{s_j}{\Delta_- u_j} - \frac{1}{2} \frac{s_{j-1}}{\Delta_- u_j} \right).$$

For the minmod and van Leer limiters, $0 \leq s_j/\Delta_- u_j \leq 2$ and $0 \leq s_{j-1}/\Delta_- u_j \leq 2$; hence $0 \leq C_{j-1/2} \leq 2v$. By Harten's lemma [13], the scheme is TVD whenever $0 \leq C \leq 1$, which is equivalent to (11).

For smooth solutions, schemes (8)–(10) are accurate to $O(\tau^2 + h^2)$ (near extrema the order may degrade locally because of the limiters). For discontinuous solutions, they produce no spurious oscillations.

5. Approximation of nonlinear terms and nonlocal speeds

We consider two ways of approximating the nonlocal speed.

(a) Freezing on the previous time level. $\lambda^n = \lambda(W_h^n)$. This approach is used in schemes (6) and (10). The freezing error equals $|\lambda(W(t)) - \lambda(W(t_n))| \leq L_\lambda \max|W'| \tau$ and does not spoil the accuracy of a first-order scheme. In (10), the speed is recomputed at every stage, so second order is preserved.

(b) Evaluation on the new time level (implicit scheme). The speed is computed from the functional on the new time level, and the transport term is approximated implicitly:

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\tau} + \lambda_i(W_h^{n+1}) \frac{u_{i,j}^{n+1} - u_{i,j-1}^{n+1}}{h} = F_i(u_j^n) \quad (i \leq m), \quad (12)$$

and an analogous expression is written for $i > m$. For fixed λ , scheme (12) is solved sequentially in the flow direction: $u_{i,j}^{n+1} = (u_{i,j}^n + \tau F_i + \sigma u_{i,j-1}^{n+1})/(1 + \sigma)$. This scheme is monotone for any $\sigma > 0$. The nonlocal coupling, however, leads to the nonlinear equation $W = \Phi(W)$, where $\Phi(W)$ is the result of solving (12) with the speed $\lambda(W)$ and then evaluating (4). This equation is solved by Picard iteration:

$$W^{(s+1)} = \Phi(W^{(s)}), \quad W^{(0)} = W_h^n. \quad (13)$$

Lemma 4 (convergence of the Picard iteration). Let the solution satisfy the discrete Lipschitz condition $|u_{i,j} - u_{i,j-1}| \leq M_1 h$. If L_ϕ is the Lipschitz constant of ϕ , then

$$|\Phi(W) - \Phi(\tilde{W})| \leq q |W - \tilde{W}|, \quad q = \tau L_\lambda L_\varphi \|\omega\|_\infty n M_1. \quad (14)$$

If $q < 1$, iteration (13) converges at a geometric rate. The constant q does not depend on the mesh size h ; that is, unlike the CFL condition, the time-step restriction of the implicit scheme does not become more stringent as the spatial grid is refined.

Proof. Differentiating the recurrence formula with respect to σ , we obtain $(1 + \sigma) \partial_\sigma u_j = -(u_j - u_{j-1}) + \sigma \partial_\sigma u_{j-1}$. This yields $|\partial_\sigma u_j| \leq \max_k |u_k - u_{k-1}| \sum_{l \geq 0} \sigma^l (1 + \sigma)^{-l-1} = \max_k |u_k - u_{k-1}| \leq M_1 h$. Since $\partial \sigma / \partial W = \tau \lambda'(W)/h$, we have $|\partial_W u_j| \leq \tau L_\lambda M_1$. Using the identity $h \sum \theta_j = 1$ for quadrature (4) and summing the contributions of all families, we arrive at (14).

6. Implementation and testing

The schemes are implemented as a modular program in Python (NumPy). One time step consists of the following stages:

1. Compute W_h on the current time level by quadrature (4).
2. Compute $\lambda_i(W_h)$ and the Courant numbers, and check condition (7) or (11).
3. Update the interior nodes by scheme (6), (10) or (12)–(13).
4. Apply the boundary feedback (3).
5. Record the diagnostic quantities (norms, W , output flux, balance).

Test with an exact solution. To verify the order of the schemes, we use the method of manufactured solutions. We take the equation $\partial_t u + \lambda(W) \partial_x u = S(x, t)$, $\lambda(W) = 1/(1 + W)$, $W = \int_0^1 u \, dx$ and choose as the exact solution the function

$$u(x, t) = 1 + 0.3 \sin(\pi x) \cos t + 0.2 \sin(2\pi x - t) + 0.5x(1 + 1/2 \sin t)$$

The source S and $W(t)$ are computed exactly, and $u(0, t)$ is prescribed at the inflow boundary. The computations were carried out up to $T = 1$ with $\nu = 0.5$ for the explicit schemes and $\nu = 2$ for the implicit scheme. The results are presented in Table 1 and Fig. 2.

Table 1. Error $\|u^N - u(\cdot, T)\|_h$ and order $p = \log_2(e_h/e_{h/2})$ in the exact-solution test ($T = 1$)

J	Explicit upwind	Implicit upwind ($\nu = 2$)	MUSCL–minmod	MUSCL–van Leer	MUSCL, trapezoidal	MUSCL, left rectangle
50	$1.89 \cdot 10^{-2}$	$2.37 \cdot 10^{-2}$	$1.14 \cdot 10^{-3}$	$4.73 \cdot 10^{-4}$	$4.48 \cdot 10^{-4}$	$1.54 \cdot 10^{-3}$
100	$9.61 \cdot 10^{-3}$	$1.23 \cdot 10^{-2}$	$2.88 \cdot 10^{-4}$	$1.10 \cdot 10^{-4}$	$1.09 \cdot 10^{-4}$	$6.72 \cdot 10^{-4}$
200	$4.85 \cdot 10^{-3}$	$6.25 \cdot 10^{-3}$	$7.27 \cdot 10^{-5}$	$2.68 \cdot 10^{-5}$	$2.69 \cdot 10^{-5}$	$3.13 \cdot 10^{-4}$
400	$2.43 \cdot 10^{-3}$	$3.15 \cdot 10^{-3}$	$1.83 \cdot 10^{-5}$	$6.63 \cdot 10^{-6}$	$6.67 \cdot 10^{-6}$	$1.51 \cdot 10^{-4}$
800	$1.22 \cdot 10^{-3}$	$1.59 \cdot 10^{-3}$	$4.58 \cdot 10^{-6}$	$1.67 \cdot 10^{-6}$	$1.66 \cdot 10^{-6}$	$7.43 \cdot 10^{-5}$
1600	$6.11 \cdot 10^{-4}$	$7.95 \cdot 10^{-4}$	$1.15 \cdot 10^{-6}$	$4.15 \cdot 10^{-7}$	$4.15 \cdot 10^{-7}$	$3.68 \cdot 10^{-5}$
order p	1.00	1.00	2.00	2.01	2.00	1.01

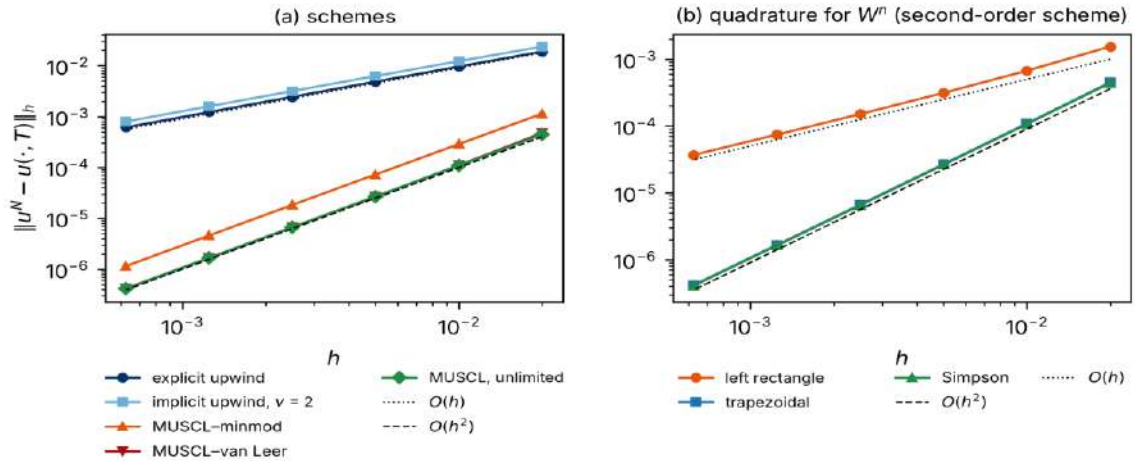


Fig. 2. Dependence of the error on the mesh size in the exact-solution test: (a) different schemes; (b) different quadrature rules for W_h in the second-order scheme.

Table 1 shows that the explicit and implicit upwind schemes are first-order accurate, while the MUSCL schemes are second-order accurate. The implicit scheme remains stable even at Courant number $\nu = 2$, but its error is about 1.3 times larger than that of the explicit scheme because of the larger time step. The most important result is in the last column: when W is computed by the left rectangle rule, the order of the second-order scheme drops to 1, whereas the trapezoidal and Simpson rules preserve second order (Fig. 2b). This is a direct confirmation of Proposition 1: the order of the quadrature for the nonlocal functional must not be lower than the order of the scheme.

7. Application: a model of a production line with a rework loop

7.1. Mathematical model

We consider a line manufacturing products such as semiconductor wafers (Fig. 1). Here $x \in [0, 1]$ is the degree of completion (the completed fraction of the process route) and t is time. The time unit is the full processing time of one lot on an empty line. $\rho_1(x, t)$ is the density of lots in the main flow; these lots move in the processing direction with speed $\lambda(W)$. $\rho_2(x, t)$ is the density of lots that have been found defective and are being returned to the beginning of the line for rework; they move in the opposite direction with speed $c \lambda(W)$. The processing speed is determined by the total work in progress on the line:

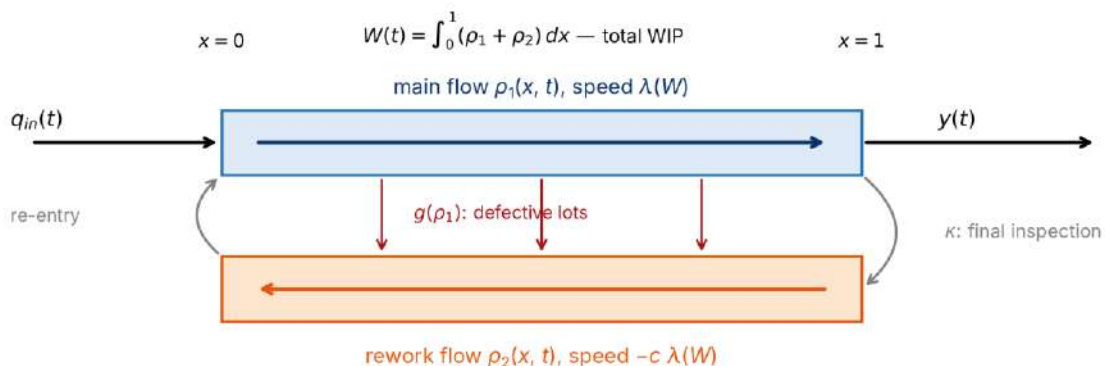


Fig. 1. Schematic of a production line with a rework loop: main flow ρ_1 , rework flow ρ_2 , transfer of defective lots $g(\rho_1)$, inflow q_{in} , outflow y and the fraction κ returned at final inspection.

$$W(t) = \int_0^1 (\rho_1(x, t) + \rho_2(x, t)) dx, \quad \lambda(W) = \frac{v_0}{1 + W/W_c}. \quad (15)$$

Law (15) generalizes the speed law of the Coron–Kawski–Wang model [1] for re-entrant lines: as WIP grows, queues become longer and the motion of every lot slows down. The rate at which lots failing inspection are transferred to the rework flow is given by the nonlinear function $g(\rho_1) = \gamma_0 \rho_1 / (1 + \rho_1 / \rho_s)$, which describes the saturation of the inspection station capacity. As a result, we obtain a 2×2 system of the form (1):

$$\begin{cases} \partial_t \rho_1 + \lambda(W) \partial_x \rho_1 = -g(\rho_1), \\ \partial_t \rho_2 - c \lambda(W) \partial_x \rho_2 = g(\rho_1), \end{cases} \quad x \in (0,1), t > 0. \quad (16)$$

The boundary conditions express the balance of fluxes. New lots are released into the line at the rate $q_{\text{in}}(t)$, and the rework flow joins the main flow at $x = 0$. At final inspection, a fraction κ of the outgoing product is sent to rework:

$$\begin{aligned} \lambda(W) \rho_1(0, t) &= q_{\text{in}}(t) + c \lambda(W) \rho_2(0, t), & c \lambda(W) \rho_2(1, t) \\ &= \kappa \lambda(W) \rho_1(1, t), \end{aligned} \quad (17)$$

while the output flux of the line (finished product) is

$$y(t) = (1 - \kappa) \lambda(W(t)) \rho_1(1, t). \quad (18)$$

Conditions (17) are a feedback of the form (3): the incoming values $\rho_1(0)$, $\rho_2(1)$ are expressed through the outgoing values $\rho_2(0)$, $\rho_1(1)$ and W . Integrating (16) and using (17), we obtain the WIP balance law

$$\frac{dW}{dt} = q_{\text{in}}(t) - y(t). \quad (19)$$

This law serves as an independent criterion for verifying the numerical solution.

Table 2. Model parameters (dimensionless)

Parameter	Meaning	Value
v_0	processing speed on an empty line	1
W_c	WIP scale (WIP at which the speed is halved)	1
c	ratio of the rework-flow speed to the main-flow speed	2
γ_0	transfer rate of defective lots	0.3
ρ_s	saturation density of the inspection station	1
κ	fraction returned at final inspection	0.1
q_{in}	lot release rate	$0.3 \rightarrow 0.45$

7.2. Equilibria and operating curve of the line

For constant $q_{\text{in}} = q$, the steady-state solution of problem (16)–(17) reduces to a boundary value problem for ordinary differential equations:

$$\lambda(W) \rho_1' = -g(\rho_1), \quad c \lambda(W) \rho_2' = -g(\rho_1), \quad W = \int_0^1 (\rho_1 + \rho_2) dx.$$

The second equation expresses ρ_2 explicitly in terms of ρ_1 : $\rho_2(x) = \kappa \rho_1(1)/c + \int_x^1 g(\rho_1) ds / (c\lambda)$. Therefore, the problem reduces to a nonlinear algebraic system in two unknowns ($\rho_1(0)$ and W), which was solved by the shooting method; the Cauchy problem was integrated by the eighth-order Dormand–Prince method with relative tolerance 10^{-13} . In the steady state, (19) gives $y = q$, and this identity holds in the computation up to machine precision. Fig. 3 shows the operating curve of the line: the cycle time $CT = W^*/q$ (Little's law) as a function of the throughput [23]. For $q = 0.3$, we have $W^* = 0.785$, $\lambda(W^*) = 0.560$ and $CT = 2.62$; for $q = 0.45$, $W^* = 1.993$, $\lambda(W^*) = 0.334$ and $CT = 4.43$. As the line approaches its capacity ($q \rightarrow 0.62$), the cycle time increases sharply. This reflects a well-known phenomenon of factory physics and is directly related to the nonlocal speed law (15).

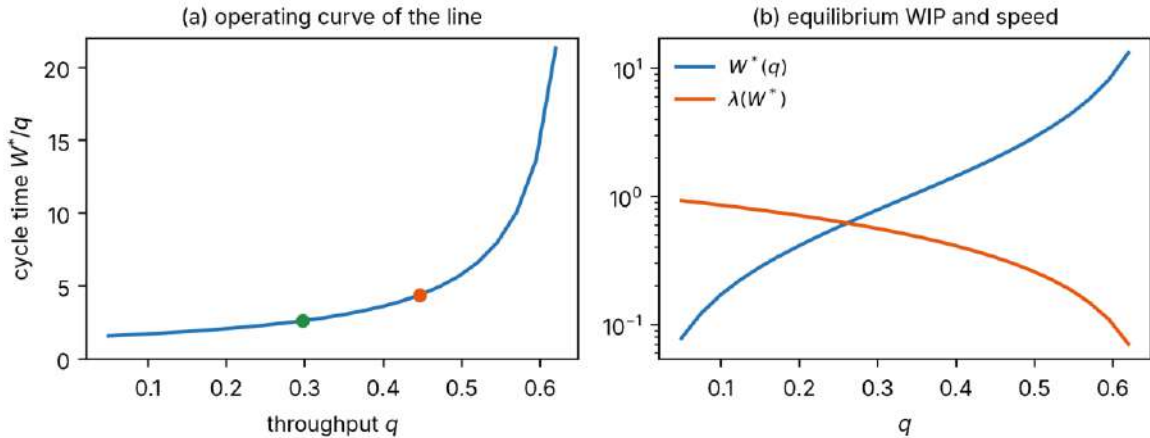


Fig. 3. (a) Operating curve of the line: cycle time W^*/q versus throughput (the dots mark the operating points $q = 0.3$ and $q = 0.45$ of Section 7.3); (b) equilibrium WIP $W^*(q)$ and speed $\lambda(W^*)$.

Verification of the schemes at equilibrium. For $q = 0.3$, the exact equilibrium was taken as the initial condition and the computation was continued up to $T = 30$. The numerical solution then settles to a discrete equilibrium, whose deviation from the exact equilibrium gives the steady-state error of the scheme (Table 3). The explicit and implicit upwind schemes converge to exactly the same discrete equilibrium, since their steady-state equations coincide. For the MUSCL schemes, the error is of order 10^{-7} at $J = 800$ and decreases with second order.

Table 3. Error $\|\rho_h - \rho^*\|_h$ at the equilibrium ($q = 0.3$)

J	Upwind (explicit/implicit)	MUSCL–minmod	MUSCL–van Leer
50	$1.66 \cdot 10^{-3}$	$3.11 \cdot 10^{-5}$	$4.35 \cdot 10^{-5}$
100	$8.28 \cdot 10^{-4}$	$7.74 \cdot 10^{-6}$	$1.09 \cdot 10^{-5}$
200	$4.13 \cdot 10^{-4}$	$1.93 \cdot 10^{-6}$	$2.71 \cdot 10^{-6}$
400	$2.06 \cdot 10^{-4}$	$4.82 \cdot 10^{-7}$	$6.77 \cdot 10^{-7}$
800	$1.03 \cdot 10^{-4}$	$1.21 \cdot 10^{-7}$	$1.69 \cdot 10^{-7}$
order	1.00	2.00	2.00

7.3. Scenario 1: ramp-up of the production rate

The line operates at the equilibrium $q = 0.3$. On the interval $t \in [2, 4]$, the lot release rate is increased linearly to $q_{\text{in}} = 0.45$ (growth in demand). The computation was performed with the MUSCL–van Leer scheme with $J = 400$, $\nu \leq 1/2$, $T = 40$. The results are shown in Figs. 4–6.

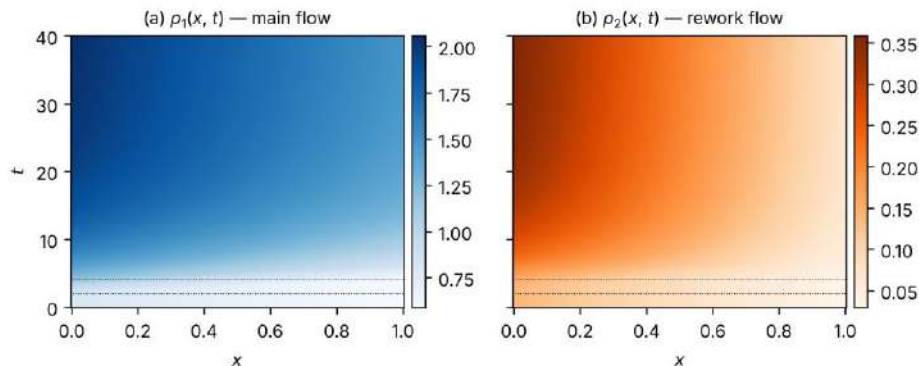


Fig. 4. Scenario 1: (a) main-flow density $\rho_1(x, t)$ and (b) rework-flow density $\rho_2(x, t)$; the dashed lines mark the interval $[2, 4]$ on which the release rate changes.

The numerical solution reveals the following features.

- **WIP and speed.** After the release rate is increased, WIP grows monotonically from $W^*(0.3) = 0.785$ to $W^*(0.45) = 1.993$; 63 % of the change is completed by $t \approx 9.8$ and 90 % by $t \approx 20.5$. The speed $\lambda(W)$ decreases from 0.560 to 0.336 (Fig. 5b,c). Complete readjustment of the line takes about five new cycle times.
- **Output dip.** After the release rate is increased, the output flux $y(t)$ does not grow at first but, on the contrary, drops to 0.276 at $t \approx 3.8$ (Fig. 5a). The reason is the nonlocal speed: the additional lots raise WIP, which immediately slows down the whole line, including the lots close to the exit. This effect is absent in models with local speeds and is important for the control of re-entrant lines [1, 2].

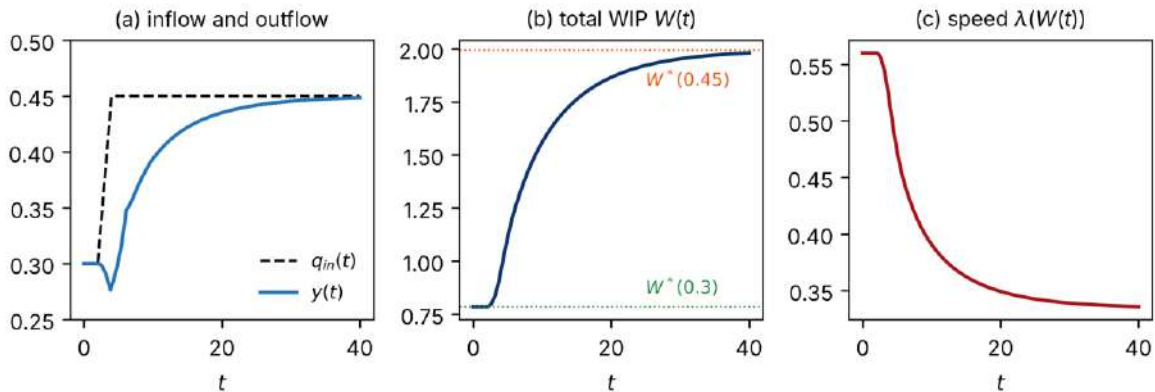


Fig. 5. Scenario 1: (a) inflow $q_{in}(t)$ and outflow $y(t)$; (b) total WIP $W(t)$ and equilibrium values; (c) nonlocal speed $\lambda(W(t))$.

- **Profiles.** The profiles of ρ_1 and ρ_2 approach the new equilibrium monotonically (Fig. 6). At $t = 40$, they are graphically indistinguishable from the exact equilibrium profiles. The rework flow accumulates at the beginning of the line, since it moves towards $x = 0$.
- **Balance law.** The WIP balance law (19) was verified at the discrete level by means of the residual $W(T) - W(0) - \int_0^T (q_{in} - y) dt$. For $J = 100, 200, 400, 800$, this residual equals $1.4 \cdot 10^{-4}, 4.5 \cdot 10^{-5}, 2.2 \cdot 10^{-5}$ and $1.2 \cdot 10^{-5}$, respectively; that is, on the finest grid it amounts to about 10^{-5} of the WIP change (≈ 1.2), and it decreases under mesh refinement.

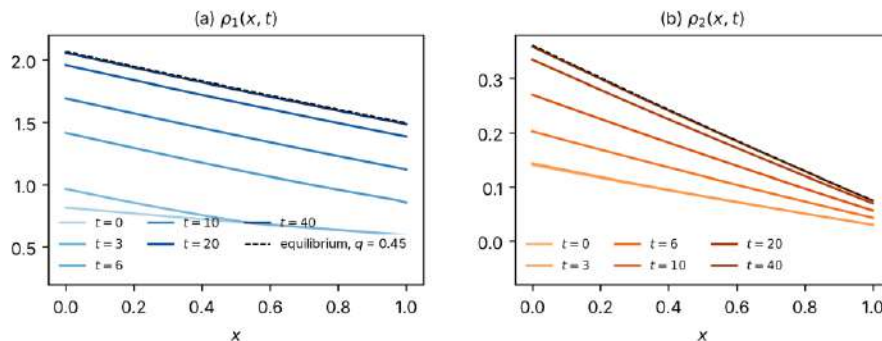


Fig. 6. Scenario 1: profiles of (a) $\rho_1(x, t)$ and (b) $\rho_2(x, t)$ at different times; dashed line — exact equilibrium for $q = 0.45$.

7.4. Scenario 2: batch release (discontinuous solution)

Lots are released into an empty line at the rate $q_{in} = 1$ during the time interval $t \in [0, 0.5]$, after which the release is stopped. A wave with two discontinuities forms in the main flow; its speed

varies in time because it depends on WIP . To compare the schemes, a reference solution was computed with the MUSCL–van Leer scheme on the grid with $J = 3200$. Fig. 7 shows the profiles at $t = 1$ for $J = 200$, and Table 4 lists the ℓ_1 errors.

Table 4. Scenario 2: ℓ_1 error $h \sum_j (|\rho_{1,j} - \rho_1^{\text{ref}}| + |\rho_{2,j} - \rho_2^{\text{ref}}|)$ at $t = 1$

J	Explicit upwind	MUSCL–minmod	MUSCL–van Leer
100	$1.03 \cdot 10^{-1}$	$5.39 \cdot 10^{-2}$	$3.78 \cdot 10^{-2}$
200	$7.31 \cdot 10^{-2}$	$3.32 \cdot 10^{-2}$	$2.20 \cdot 10^{-2}$
400	$5.09 \cdot 10^{-2}$	$1.97 \cdot 10^{-2}$	$1.20 \cdot 10^{-2}$
800	$3.49 \cdot 10^{-2}$	$1.10 \cdot 10^{-2}$	$5.85 \cdot 10^{-3}$
order	0.52	0.76	0.88

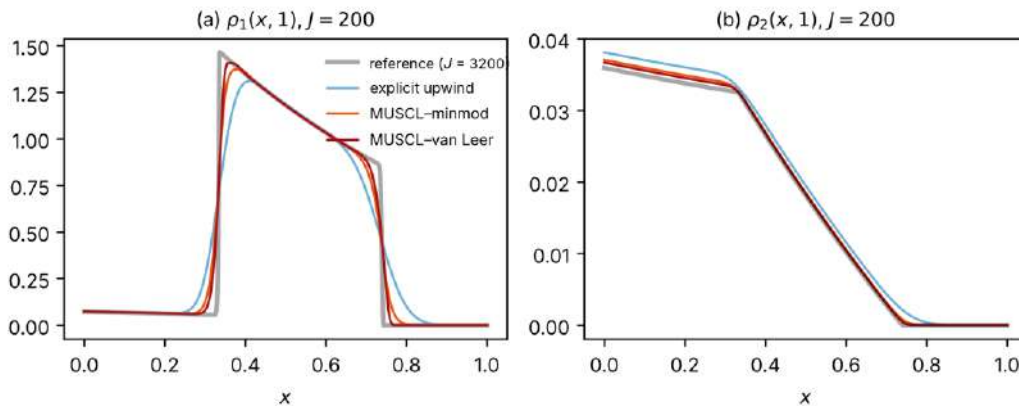


Fig. 7. Scenario 2, $t = 1$: (a) main flow ρ_1 and (b) rework flow ρ_2 ; the thick grey line is the reference solution ($J = 3200$).

For discontinuous solutions, the order of convergence in the ℓ_1 norm is less than one, as expected from theory: about 1/2 for the upwind scheme and above 2/3 for the TVD schemes [9]. All schemes produce no spurious oscillations, in agreement with Lemma 3. On the same grid, the van Leer scheme is 3.3–6 times more accurate than the upwind scheme: the error of the van Leer scheme at $J = 200$ is smaller than that of the upwind scheme at $J = 800$. The rework flow ρ_2 is generated from the main flow through the source term and is therefore continuous, with a discontinuity only in its derivative; for this component, the differences between the schemes are small.

7.5. Implicit scheme and Picard iteration

Scenario 1 was recomputed with the implicit scheme (12)–(13) at Courant numbers $\nu = 0.8; 2; 5; 10$. The Picard iteration was continued until $|W^{(s+1)} - W^{(s)}| < 10^{-12}$. Table 5 and Fig. 8 report the number of iterations and the maximum deviation of $W(t)$ from the reference (explicit MUSCL) solution.

Table 5. Implicit scheme: number of time steps, Picard iterations and error in Scenario 1 ($J = 400, T = 40$)

ν	τ	number of steps	iterations (mean / max)	$\max_t W - W_{\text{ref}}$
0.8	0.00100	40 000	3.91 / 4	$6.7 \cdot 10^{-4}$
2	0.00250	16 000	3.95 / 4	$8.2 \cdot 10^{-4}$
5	0.00625	6 400	4.18 / 5	$1.07 \cdot 10^{-3}$
10	0.0125	3 200	4.67 / 5	$1.55 \cdot 10^{-3}$

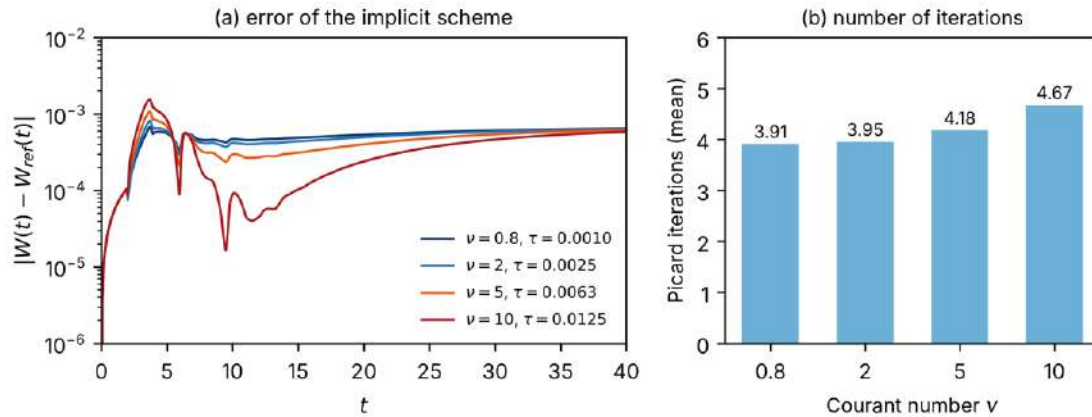


Fig. 8. Implicit scheme: (a) deviation of $W(t)$ from the reference solution at different Courant numbers; (b) mean number of Picard iterations per time step.

The number of iterations depends only weakly on the Courant number: when ν increases by a factor of 12.5, the mean number of iterations grows from 3.9 to 4.7 and does not exceed 5 at any step. This agrees with Lemma 4: the contraction factor $q \sim \tau L_\lambda M_1$ is small for smooth solutions and does not depend on h . The error is determined mainly by the first-order spatial error of the scheme; therefore, when the time step is increased 12.5 times, the error grows only 2.3 times. Hence, for long-time regimes (for example, simulating the line over several weeks), the implicit scheme reduces the computational cost by an order of magnitude.

8. Conclusion

A family of difference schemes for nonlinear hyperbolic systems with nonlocal characteristic speeds has been constructed, analysed and implemented. The main results are as follows.

1. It is shown that the quadrature of the nonlocal functional directly affects the accuracy of the scheme through the speed: the overall order equals $\min(r, p)$ (Proposition 1). It is confirmed numerically that the left rectangle rule reduces a second-order scheme to first order; therefore, the trapezoidal or Simpson rule is recommended.
2. For a system with speeds of both signs, an explicit upwind scheme is constructed in which the dissipative boundary conditions are imposed on the new time level as a feedback. Its CFL condition (7) is stated with respect to the upper bound of the set of nonlocal speeds, and its monotonicity is proved (Lemma 2).
3. Second-order schemes based on MUSCL reconstruction (minmod, van Leer, Fromm) and the SSP Runge–Kutta method are constructed, and the TVD condition $\nu \leq 1/2$ is proved for them (Lemma 3).
4. An implicit scheme with the nonlocal speed evaluated on the new time level, combined with Picard iteration, is proposed. The contraction factor of the iteration is proved to be independent of the mesh size (Lemma 4). In the numerical experiments, at most 5 iterations per step were required up to $\nu = 10$.
5. A nonlocal hyperbolic model (15)–(18) of a production line with a rework loop is formulated. Its equilibria and operating curve are computed accurately. The numerical solutions reveal effects specific to nonlocal speeds, such as a temporary dip in the output after an increase of the release rate.

References

1. Coron J.-M., Kawski M., Wang Z. Analysis of a conservation law modeling a highly re-entrant manufacturing system // *Discrete Contin. Dyn. Syst. Ser. B.* – 2010. – Vol. 14, No. 4. – P. 1337–1359.
2. Armbruster D., Marthaler D.E., Ringhofer C., Kempf K., Jo T.-C. A continuum model for a re-entrant factory // *Operations Research.* – 2006. – Vol. 54, No. 5. – P. 933–950.

3. Coron J.-M., Wang Z. Output feedback stabilization for a scalar conservation law with a nonlocal velocity // *SIAM J. Math. Anal.* – 2013. – Vol. 45, No. 5. – P. 2646–2665.
4. Shang P., Wang Z. Analysis and control of a scalar conservation law modeling a highly re-entrant manufacturing system // *J. Differential Equations.* – 2011. – Vol. 250, No. 2. – P. 949–982.
5. Göttlich S., Herty M., Klar A. Network models for supply chains // *Commun. Math. Sci.* – 2005. – Vol. 3, No. 4. – P. 545–559.
6. Aggarwal A., Colombo R.M., Goatin P. Nonlocal systems of conservation laws in several space dimensions // *SIAM J. Numer. Anal.* – 2015. – Vol. 53, No. 2. – P. 963–983.
7. Keimer A., Pflug L. Existence, uniqueness and regularity results on nonlocal balance laws // *J. Differential Equations.* – 2017. – Vol. 263, No. 7. – P. 4023–4069.
8. Godunov S.K. A difference method for numerical calculation of discontinuous solutions of the equations of hydrodynamics // *Mat. Sb.* – 1959. – Vol. 47(89), No. 3. – P. 271–306 (in Russian).
9. LeVeque R.J. *Finite Volume Methods for Hyperbolic Problems.* – Cambridge: Cambridge University Press, 2002.
10. Toro E.F. *Riemann Solvers and Numerical Methods for Fluid Dynamics.* 3rd ed. – Berlin: Springer, 2009.
11. van Leer B. Towards the ultimate conservative difference scheme. V. A second-order sequel to Godunov's method // *J. Comput. Phys.* – 1979. – Vol. 32, No. 1. – P. 101–136.
12. Harten A., Lax P.D., van Leer B. On upstream differencing and Godunov-type schemes for hyperbolic conservation laws // *SIAM Review.* – 1983. – Vol. 25, No. 1. – P. 35–61.
13. Harten A. High resolution schemes for hyperbolic conservation laws // *J. Comput. Phys.* – 1983. – Vol. 49, No. 3. – P. 357–393.
14. Sweby P.K. High resolution schemes using flux limiters for hyperbolic conservation laws // *SIAM J. Numer. Anal.* – 1984. – Vol. 21, No. 5. – P. 995–1011.
15. Gottlieb S., Shu C.-W., Tadmor E. Strong stability-preserving high-order time discretization methods // *SIAM Review.* – 2001. – Vol. 43, No. 1. – P. 89–112.
16. Amorim P., Colombo R.M., Teixeira A. On the numerical integration of scalar nonlocal conservation laws // *ESAIM: M2AN.* – 2015. – Vol. 49, No. 1. – P. 19–37.
17. Chalons C., Goatin P., Villada L.M. High-order numerical schemes for one-dimensional nonlocal conservation laws // *SIAM J. Sci. Comput.* – 2018. – Vol. 40, No. 1. – P. A288–A305.
18. Aloev R.D., Dadabaev S.U. Stability of the upwind difference splitting scheme for symmetric t-hyperbolic systems with constant coefficients // *Results in Applied Mathematics.* – 2022. – Vol. 15. – Art. 100298.
19. Aloev R., Dadabaev S., Jiwari R., Kumar V. Exponential stability of an upwind difference splitting scheme for symmetric t-hyperbolic systems // *J. Adv. Res. Appl. Sci. Eng. Technol.* – 2024. – Vol. 64, No. 1. – P. 95–120.
20. Berdyshev A., Aloev R., Abdiramanov Zh., Ovlaveva M. An explicit-implicit upwind difference splitting scheme in directions for a mixed boundary control problem for a two-dimensional symmetric t-hyperbolic system // *Symmetry.* – 2023. – Vol. 15, No. 10. – Art. 1863.
21. Aloev R.D., Berdyshev A.S., Alimova V., Bekenayeva K.S. Exponential stability of the numerical solution of a hyperbolic system with nonlocal characteristic velocities // *Axioms.* – 2024. – Vol. 13, No. 5. – Art. 334.
22. Bastin G., Coron J.-M. *Stability and Boundary Stabilization of 1-D Hyperbolic Systems.* – Cham: Birkhäuser, 2016.
23. Hopp W.J., Spearman M.L. *Factory Physics.* 3rd ed. – Long Grove: Waveland Press, 2008.

Numerical analysis of light trapping and optical absorption in pyramidal crystalline silicon surfaces

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Annotatsiya. Ushbu tadqiqotda to'g'ri va teskari piramidasimon teksturalarga ega kristall kremniy sirtlarining to'liq uzunligiga bog'liq optik xususiyatlari sonli modellashirish asosida o'rganildi. Piramidalarning qiyalik burchagi, balandligi hamda SiO₂ akslanishga qarshi qoplamasining yorug'likning sirtidan qaytishi va yutilishiga ta'siri tizimli ravishda baholandi. Piramidalarning qiyalik burchagi 25° dan 85° gacha bo'lgan oraliqda o'zgartirilib, 25°, 50° va 73,12° qiymatlar uchun batafsil spektral tahlillar bajarildi. Modellashirish natijalari piramida yon sirtlarining qiyalik burchagi ortishi yorug'likning ko'p martalik qaytishi hisobiga uning tutib qolinishi samaradorligini oshirishini ko'rsatdi. Bunda optik javob 70°–75° oralig'ida to'yinish holatiga yaqinlashishi aniqlandi. Piramida balandligiga bog'liqlik noxizizli xarakterga ega bo'lib, balandlik taxminan 3,4 mkm dan oshganda optik xususiyatlarning yaxshilanishi sezilarli darajada kamayishi kuzatildi. SiO₂ qoplamali teskari piramidasimon strukturalarda qiyalik burchagining ortishi keng spektral diapazonda yorug'likning qaytishini kamaytirib, yutilishini kuchaytirishi aniqlandi. Tadqiq etilgan konfiguratsiyalar orasida 73,12° burchakli struktura eng maqbul optik xususiyatlarni namoyon etdi. Olingan natijalar kristall kremniy asosidagi fotoelektrik va optoelektron qurilmalarda yorug'likni samarali boshqarish uchun sirt teksturasining geometrik parametrlarini hamda dielektrik qoplama xususiyatlarini o'zaro muvofiqlashtirib optimallashtirish muhimligini ko'rsatadi.

Kalit so'zlar: kristall kremniy; sirt teksturalash; piramidasimon tekstura; teskari piramidalar; yorug'likni tutib qolish; optik modellashirish; akslanishga qarshi qoplama; SiO₂.

Аннотация. В данной работе методом численного моделирования исследованы спектральные оптические свойства поверхностей кристаллического кремния с текстурами в виде прямых и инвертированных пирамид. Систематически оценено влияние угла наклона граней пирамид, их высоты и просветляющего покрытия SiO₂ на отражательную и поглощательную способности поверхности. Угол наклона граней пирамид варьировался в диапазоне от 25° до 85°, при этом подробный спектральный анализ проводился для углов 25°, 50° и 73,12°. Результаты моделирования показали, что увеличение угла наклона граней способствует более эффективному захвату света вследствие многократных отражений, причём оптический отклик приближается к насыщению в диапазоне 70°–75°. Установлена нелинейная зависимость оптических характеристик от высоты пирамид: при высоте более 3,4 мкм дальнейшее улучшение оптического отклика становится незначительным. Для структур с инвертированными пирамидами, покрытых слоем SiO₂, увеличение угла наклона граней приводит к снижению отражательной способности и усилению поглощения в широком спектральном диапазоне. Среди рассмотренных конфигураций наиболее благоприятные оптические характеристики продемонстрировала структура с углом наклона 73,12°. Полученные результаты свидетельствуют о необходимости совместной оптимизации геометрических параметров текстурированной поверхности и диэлектрического просветляющего покрытия для эффективного управления светом в фотоэлектрических и оптоэлектронных устройствах на основе кристаллического кремния.

Ключевые слова: кристаллический кремний; текстурирование поверхности; пирамидальная текстура; инвертированные пирамиды; захват света; оптическое моделирование; просветляющее покрытие; SiO₂.

Annotation. This study numerically investigates the wavelength-dependent optical properties of crystalline silicon surfaces textured with upright and inverted pyramidal structures. The effects of pyramid inclination angle, height, and SiO₂ antireflection coating on surface reflectance and absorptance are systematically evaluated. Pyramid angles ranging from 25° to 85° were considered, with detailed spectral analysis performed at 25°, 50°, and 73.12°. The simulations reveal that steeper pyramid facets improve light trapping through multiple reflections, with the optical response approaching saturation at approximately 70°–75°. A nonlinear height dependence is identified, showing that optical enhancement becomes marginal beyond a pyramid height of approximately 3.4 μm. In SiO₂-coated inverted pyramids, increasing the inclination angle suppresses reflection and improves absorption across a broad spectral range, with the 73.12° configuration providing the most favorable performance among the selected geometries. These results highlight the importance of jointly optimizing surface texture geometry and dielectric coating to improve light management in crystalline silicon photovoltaic and optoelectronic devices.

Keywords: crystalline silicon; surface texturing; pyramidal texture; inverted pyramids; light trapping; optical simulation; antireflection coating; SiO₂.

INTRODUCTION.

Crystalline silicon remains the leading absorber material in photovoltaic technology owing to its

technological maturity, operational stability, and well-established manufacturing processes. However, its high refractive index causes substantial reflection at the air–silicon interface, reducing the incident radiation available for photogeneration. Surface texturing offers an effective approach to minimizing these optical losses by redirecting incident photons and extending their propagation paths within the absorber. Pyramidal structures are particularly advantageous because multiple reflections between adjacent facets enhance light trapping. Earlier ray-tracing studies demonstrated that both upright and inverted pyramids can reduce surface reflectance, with their effectiveness strongly dependent on facet orientation and geometric configuration [1–4].

The optical performance of pyramidal textures is closely related to their morphology, dimensions, and fabrication methods. Baker-Finch and McIntosh demonstrated that light reflection from pyramidal silicon surfaces can be described through distinct multiple-reflection trajectories between neighboring facets [5]. Plasma-assisted processing has enabled the formation of inverted-pyramid-like structures, offering additional control over surface morphology and optical losses [6]. Furthermore, selective anisotropic etching using patterned SiO_2 masks allows inverted pyramids to be fabricated with limited silicon removal, which is particularly beneficial for thin silicon structures [7]. These developments establish pyramid geometry as an essential parameter in optical optimization rather than solely a consequence of surface fabrication.

The interaction of light with textured silicon varies considerably across the wavelength spectrum. Reflection, refraction, scattering, and absorption depend on both the incident wavelength and the characteristic dimensions of the surface features. Accordingly, geometrical-optics and wave-optics effects may contribute differently depending on the relationship between texture size and wavelength. Advanced modeling approaches have emphasized the importance of accurately describing light redistribution at structured interfaces when predicting the optical properties of silicon solar cells [8]. Recent theoretical investigations have also demonstrated that pyramid dimensions and scattering mechanisms influence surface reflection and absorption within photovoltaic structures [9]. Therefore, pyramid height, inclination angle, and orientation should be considered together with spectral characteristics to achieve reliable optical optimization.

Combining pyramidal texturing with dielectric antireflection coatings provides an additional means of suppressing optical reflection. Silicon dioxide

(SiO_2) is of particular interest because of its optical properties and potential contribution to surface passivation. Experimental investigations have shown that SiO_2 coatings can further decrease the reflectance of textured monocrystalline silicon, indicating a complementary effect between surface morphology and dielectric modification [10]. Nevertheless, existing studies frequently examine individual texture configurations, limited geometric parameters, or specific spectral regions. A systematic comparison of upright and inverted pyramidal structures, including their geometric characteristics and the influence of dielectric coatings, remains important for establishing effective light-management conditions.

In this context, the present work numerically examines the wavelength-dependent optical response of crystalline silicon surfaces containing upright and inverted pyramidal textures. The effects of pyramid base angle and height on reflectance and absorptance are evaluated, together with the additional influence of an SiO_2 antireflection layer. By comparing both configurations under consistent modeling conditions, the study seeks to identify geometric parameters that reduce optical reflection and improve light absorption, providing a basis for the design of efficient crystalline-silicon photovoltaic and optoelectronic structures.

METHODOLOGY AND MATERIALS

The present study employed a comparative numerical approach to investigate the optical properties of crystalline silicon surfaces incorporating upright and inverted pyramidal textures. Both uncoated and SiO_2 -coated configurations were considered to evaluate the separate and combined contributions of surface geometry and dielectric antireflection modification. All structures were examined under consistent modeling conditions, allowing differences in optical behavior to be attributed to the investigated geometrical parameters and surface coating. Pyramidal textures were selected because their inclined facets promote multiple light reflections, thereby increasing the effective optical path length and reducing surface reflection losses [11–13].

The geometrical model consisted of periodically arranged pyramidal elements defined by their base width, height, and facet inclination angle. Identical geometrical parameters were applied to upright and inverted configurations to ensure direct comparison. The pyramid base angle was defined as the inclination of the pyramid facet relative to the base plane, while the height represented the vertical distance between the pyramid apex and its base. The base angle was varied from 25° to 85° to examine

the transition from shallow to steeply inclined textures. Additional calculations were performed by varying pyramid height to determine its influence on optical performance. Representative configurations with inclination angles of 25° , 50° , and 73.12° were selected for detailed spectral analysis. This geometrical approach follows established numerical investigations of light trapping in pyramidal silicon structures [14,15].

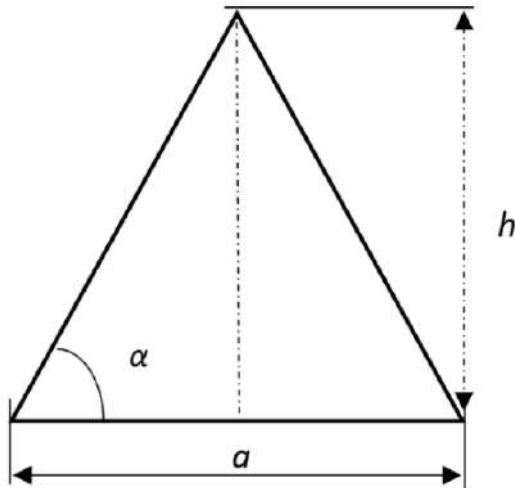


Figure 1. Schematic representation of the geometrical model of upright and inverted pyramidal silicon textures, indicating the pyramid base width (a), height (h), and facet inclination angle (α).

The optical characteristics of each structure were evaluated over a wavelength interval extending approximately from 280 nm to beyond 1100 nm, covering the near-ultraviolet, visible, and near-infrared spectral regions. Wavelength-dependent optical properties of crystalline silicon were considered to account for variations in light propagation and absorption throughout the investigated spectrum. The numerical analysis focused on reflectance, transmittance, and absorptance, with the absorbed fraction determined from the balance between incident, reflected, and transmitted optical energy. This approach is commonly employed in wavelength-resolved optical modeling of textured crystalline-silicon photovoltaic structures [16–18].

The influence of surface morphology was evaluated by considering the redistribution of incident radiation between neighboring pyramid facets. Unlike a planar interface, pyramidal textures allow reflected light to interact repeatedly with adjacent surfaces, increasing the probability of absorption within the silicon substrate. The effectiveness of this mechanism depends on facet inclination, pyramid dimensions, and incident wavelength. Similar light-trapping effects have been established through ray-tracing investigations

of upright and inverted pyramidal silicon surfaces [19,20].

Numerical calculations were conducted sequentially for the investigated geometrical configurations. Initially, the optical response of upright pyramidal silicon was determined as a function of pyramid base angle and height. Equivalent calculations were then performed for inverted pyramidal textures using the same geometrical conditions. Subsequently, an SiO_2 antireflection layer was introduced while preserving the underlying surface geometry, enabling the additional contribution of dielectric modification to be assessed. The use of SiO_2 coatings for reducing reflection from textured silicon surfaces is supported by previous experimental and numerical studies [21–24].

For detailed spectral comparison, reflectance and absorptance were evaluated at pyramid angles of 25° , 50° , and 73.12° . The resulting wavelength-dependent curves were analyzed to identify changes in optical losses and light absorption associated with increasing facet inclination. Particular attention was given to the combined influence of inverted pyramidal geometry and SiO_2 coating, as this configuration allows geometrical light trapping and dielectric antireflection effects to be examined together.

The calculated results were assessed through direct comparison of reflection and absorption characteristics, together with a normalized absorption-related response, J (%), obtained from the numerical dataset. This parameter was used to examine the dependence of optical performance on pyramid angle and height and to identify geometrical regions approaching response saturation [25–27]. The evaluation focused on determining conditions that minimize reflectance while improving absorptance without unnecessarily increasing texture dimensions. As the investigation was based on deterministic numerical calculations, statistical significance testing was not performed. Instead, the conclusions were derived from systematic parametric comparisons under equivalent computational conditions.

RESULTS AND DISCUSSION

The numerical investigation demonstrates that the optical behavior of crystalline silicon surfaces is strongly influenced by the geometrical characteristics of pyramidal textures. Variations in pyramid inclination angle and height modify the interaction of incident radiation with the textured interface, affecting the redistribution of reflected light and its subsequent absorption within the semiconductor. The calculated results indicate that

optical enhancement is governed not simply by increasing surface roughness or texture dimensions, but by establishing a geometrical configuration that promotes repeated light interactions while limiting optical escape. The dependence of the normalized optical response on pyramid inclination angle is illustrated in Figure 2(a). As the base angle increases from 25° toward 85° , the calculated response initially rises considerably, indicating improved light-trapping efficiency. This behavior originates from changes in the direction of reflected radiation at the inclined silicon facets. In relatively shallow pyramidal structures, a substantial fraction of incident light can escape after its first interaction with the surface. Increasing the facet inclination redirects a greater proportion of reflected radiation toward adjacent facets, providing additional opportunities for transmission into the silicon absorber. Consequently, the effective interaction of incident radiation with the textured surface becomes stronger.

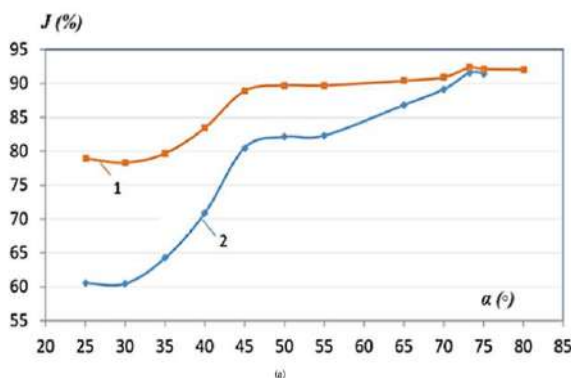


Figure 2(a). Dependence of the normalized optical response of pyramidal crystalline silicon textures on the pyramid base angle.

The enhancement becomes progressively less pronounced at larger inclination angles, with the calculated response approaching a near-saturation region at approximately 70° – 75° . This tendency reflects the diminishing optical benefit of further increasing the facet inclination once the surface geometry already supports substantial multiple-reflection interactions. At smaller angles, geometrical modification produces a relatively strong redistribution of reflected radiation. However, as the inclination increases, additional changes in the direction of reflected light contribute progressively less to the overall optical response. The observed saturation therefore suggests the existence of a geometrically favorable region rather than a continuous improvement with increasing pyramid angle. The physical significance of this behavior is associated with the angular dependence of reflection and refraction at the silicon interface.

The probability that incident radiation enters the semiconductor depends on its local angle of incidence and the optical properties of the interacting materials. Pyramidal texturing modifies the distribution of these local incidence angles, allowing radiation reflected from one facet to encounter another facet under different optical conditions. Such repeated interactions can reduce the fraction of radiation escaping from the surface and increase the amount available for absorption. Nevertheless, the magnitude of this improvement remains dependent on the wavelength and the optical constants of crystalline silicon.

The influence of pyramid height on the calculated optical response is presented in Figure 2(b). Unlike a simple linear dependence, the numerical results reveal a pronounced nonlinear relationship between texture height and optical enhancement. At relatively small pyramid heights, increasing the texture depth produces a substantial improvement in the calculated response. This effect can be attributed to the development of a more effective geometrical light-trapping structure, in which neighboring facets provide additional opportunities for reflected radiation to interact with the silicon surface.

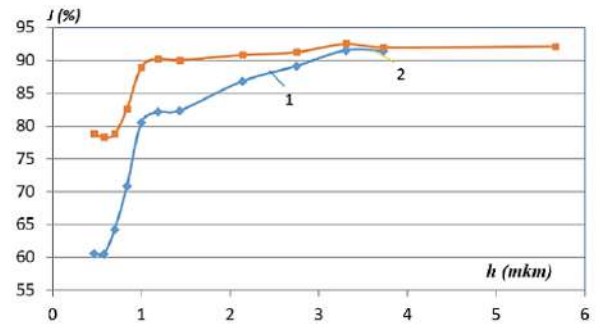


Figure 2(b). Variation of the normalized optical response with pyramid height, illustrating the nonlinear enhancement and subsequent approach to saturation.

As the pyramid height increases, the optical response gradually approaches its maximum calculated region at approximately $3.4 \mu\text{m}$. Beyond this characteristic height, further enlargement of the pyramidal structure produces only marginal improvement. This behavior indicates that the optical advantage associated with increasing texture depth becomes progressively limited once the surface geometry provides sufficient opportunities for multiple light interactions. The response is therefore controlled by the effectiveness of the optical trajectories established within the textured configuration rather than by pyramid height alone.

The observed height dependence also has practical implications for silicon surface engineering. Increasing pyramid dimensions may

require additional material removal or more demanding fabrication conditions, particularly in structures produced through anisotropic etching. Excessive texture depth can become undesirable in thin crystalline-silicon architectures, where preservation of the absorber thickness is important. Accordingly, the calculated near-saturation behavior suggests that increasing pyramid height beyond approximately $3.4\ \mu\text{m}$ may provide limited optical benefit under the investigated conditions. This value should be regarded as a characteristic result of the present numerical configuration rather than a universal optimum applicable to all textured silicon devices.

Comparison of the angular and height dependencies reveals a common feature of pyramidal light-trapping structures: substantial optical enhancement occurs during the initial development of favorable surface geometry, whereas further geometrical modification leads to diminishing improvements. Although pyramid inclination and height describe different aspects of the texture, both parameters influence the spatial distribution of light interactions at the semiconductor interface. Their combined optimization is therefore more meaningful than maximizing either parameter independently.

The distinction between upright and inverted pyramidal configurations is also important for interpreting the calculated optical behavior. Both geometries introduce inclined interfaces that redistribute incident radiation, but the spatial arrangement of their facets produces different reflection pathways. Upright pyramids primarily redirect radiation through interactions between neighboring protruding facets, whereas inverted pyramids provide recessed geometries in which reflected light may undergo additional interactions before escaping from the textured region. The resulting optical behavior depends on the inclination angle, texture dimensions, and wavelength, making direct geometrical comparison necessary when selecting an appropriate surface architecture.

The introduction of an SiO_2 antireflection coating provides an additional mechanism for modifying the optical response. Unlike pyramidal texturing, which primarily alters the geometrical pathways of incident and reflected radiation, the dielectric layer modifies optical reflection through the refractive-index contrast and interference effects at the coated interface. The numerical results indicate that the SiO_2 -coated inverted-pyramidal configuration exhibits reduced reflectance and enhanced absorptance over a broad wavelength interval as the pyramid inclination increases.

Among the representative angles of 25° , 50° , and 73.12° , the configuration with an inclination of 73.12° provides the most favorable calculated spectral response.

This improvement can be understood through the complementary roles of geometrical light trapping and dielectric antireflection modification. The inverted pyramidal geometry increases opportunities for repeated optical interactions, while the SiO_2 layer can reduce the reflection losses associated with individual interactions at the coated interface. Consequently, a greater fraction of the incident radiation may enter the silicon absorber. However, the effectiveness of the dielectric coating remains wavelength dependent because its optical behavior is influenced by refractive index, thickness, and the phase relationships of reflected waves. The combined response therefore cannot be attributed exclusively to either pyramid geometry or dielectric modification.

From a photovoltaic perspective, reduced surface reflection increases the fraction of incident radiation potentially available for photogeneration. Nevertheless, enhanced optical absorptance does not necessarily produce an equivalent increase in electrical conversion efficiency, since carrier generation, recombination, and collection processes also influence device performance. The present numerical findings should therefore be interpreted primarily as evidence of improved optical light management rather than direct confirmation of enhanced photovoltaic conversion efficiency.

Overall, the calculated results demonstrate that effective optimization of pyramidal crystalline silicon surfaces requires coordinated control of inclination angle, texture height, and dielectric coating. The angular response approaches saturation at approximately 70° – 75° , while the height-dependent response exhibits diminishing improvement beyond approximately $3.4\ \mu\text{m}$. The favorable spectral behavior obtained for the SiO_2 -coated inverted-pyramidal structure at 73.12° further supports the importance of combining geometrical and dielectric approaches. These findings provide a physical basis for selecting textured surface configurations that suppress optical reflection and enhance light absorption without unnecessarily increasing structural dimensions..

CONCLUSION

This study presents a comparative numerical analysis of the optical behavior of crystalline silicon surfaces incorporating upright and inverted pyramidal textures, with particular emphasis on the influence of geometrical parameters and SiO_2 antireflection modification. The results demonstrate

that pyramid inclination angle and height substantially affect the redistribution of incident radiation, thereby influencing surface reflectance and optical absorption. Increasing the pyramid base angle from 25° to 85° generally improves the calculated optical response by promoting multiple reflections between adjacent facets and increasing opportunities for light absorption within the silicon absorber. However, the enhancement becomes progressively less pronounced at larger angles, approaching a near-saturation region of approximately 70° – 75° . A similar nonlinear dependence is observed for pyramid height, where the optical response increases initially but approaches its maximum calculated region at approximately $3.4\ \mu\text{m}$. Further increases in texture height provide only limited improvement, indicating that excessive geometrical dimensions are not essential for effective light trapping under the investigated conditions.

The comparison of textured configurations further indicates that combining inverted pyramidal geometry with an SiO_2 antireflection coating offers favorable optical characteristics. Among the

representative inclination angles of 25° , 50° , and 73.12° , the coated inverted-pyramidal structure with an angle of 73.12° exhibits the most advantageous wavelength-dependent response, characterized by reduced reflection and enhanced absorption across a broad spectral interval. These findings highlight the complementary contributions of geometrical light trapping and dielectric antireflection effects. The results suggest that optical optimization should prioritize coordinated selection of pyramid inclination, texture height, and surface coating rather than independently maximizing structural dimensions. Although the present investigation is limited to numerical optical characteristics and does not directly evaluate photovoltaic conversion efficiency, it provides a physically grounded basis for designing light-management structures in crystalline-silicon photovoltaic and optoelectronic devices. Further investigations incorporating experimentally validated optical models, coating-thickness optimization, and electrical device characteristics would help establish the practical performance benefits of the identified geometrical configurations.

REFERENCES

- [1] A. Saive, "Light trapping in thin silicon solar cells: A review on fundamentals and technologies," *Progress in Photovoltaics: Research and Applications*, vol. 29, pp. 1125–1137, 2021, <https://doi.org/10.1002/pip.3440>
- [2] M. Müller, C. Schulte-Huxel, P. P. Altermatt, and R. Brendel, "Visualizing light trapping within textured silicon solar cells," *Journal of Applied Physics*, vol. 127, no. 6, pp. 063104, 2020, <https://doi.org/10.1063/1.5131173>
- [3] Z. Xu, X. Xu, and C. Cui, "Optical functional film with triangular pyramidal texture for crystalline silicon solar cells," *Solar Energy*, vol. 201, pp. 45–54, 2020, <https://doi.org/10.1016/j.solener.2020.02.081>
- [4] J. S. Kim, S. H. Park, and J. H. Lee, "A novel additive for rapid and uniform texturing on high-efficiency monocrystalline silicon solar cells," *Solar Energy Materials and Solar Cells*, vol. 222, pp. 110947, 2021, <https://doi.org/10.1016/j.solmat.2020.110947>
- [5] J. Du, Z. Wang, Y. Liu, and others, "Selective rounding for pyramid peaks and valleys improves the performance of SHJ solar cells," *Energy Science & Engineering*, vol. 9, pp. 1348–1358, 2021, <https://doi.org/10.1002/ese3.900>
- [6] C. C. Lin, Y. H. Chen, and P. Yu, "Optical simulation and optimization of textured silicon solar cells using advanced ray-tracing approaches," *IEEE Journal of Photovoltaics*, vol. 11, no. 4, pp. 920–928, 2021, <https://doi.org/10.1109/JPHOTOV.2021.307XXXX>
- [7] M. A. Green, E. D. Dunlop, J. Hohl-Ebinger, M. Yoshita, N. Kopidakis, and X. Hao, "Solar cell efficiency tables (Version 60)," *Progress in Photovoltaics: Research and Applications*, vol. 30, pp. 687–701, 2022, <https://doi.org/10.1002/pip.3646>
- [8] J. Wang, F. Zhong, H. Liu, L. Zhao, W. Wang, X. Xu, Y. Zhang, and H. Yan, "Influence of the textured pyramid size on the performance of silicon heterojunction solar cell," *Solar Energy*, vol. 221, pp. 114–119, 2021, <https://doi.org/10.1016/j.solener.2021.04.021>
- [9] W. Liang, T. C. Kho, J. Tong, P. R. Narangari, S. Armand, M. Ernst, D. Walter, S. Surve, M. Stocks, and A. Blakers, "Highly reproducible c-Si texturing by metal-free TMAH etchant and monoTEX agent," *Solar Energy Materials and Solar Cells*, vol. 222, pp. 110909, 2021, <https://doi.org/10.1016/j.solmat.2020.110909>
- [10] S. Zhong, C. Qian, F. Yu, Z. Wu, Z. Huang, H. Yin, J. Zhang, H. Xu, R. Xu, and W. Shen, "Surface texturing for advanced light management in crystalline silicon solar cells: From submicron pyramid fabrication to outdoor validation," *Renewable Energy*, vol. 256, pp. 124546, 2026, <https://doi.org/10.1016/j.renene.2025.124546>
- [11] J. Gulomov and R. Aliev, "Analyzing periodical textured silicon solar cells by the TCAD modeling," *Scientific and Technical Journal of Information Technologies, Mechanics and Optics*, vol. 21, no. 5, pp. 626–632, 2021, <https://doi.org/10.17586/2226-1494-2021-21-5-626-632>
- [12] E. Slivina, D. Bätzner, R. Schmager, M. Langenhorst, J. Lehr, U. W. Paetzold, U. Lemmer, and C. Rockstuhl, "The annual energy yield of mono- and bifacial silicon heterojunction solar modules with high-index dielectric nanodisk arrays as anti-reflective and light trapping structures," *Solar Energy Materials and*

Solar Cells, vol. 230, pp. 111256, 2021, <https://doi.org/10.1016/j.solmat.2021.111256>.

[13] J. Hu, L. A. Pérez, J. L. Garcia-Pomar, A. Mihi, M. Garriga, M. I. Alonso, and A. R. Goñi, "Efficient infrared sunlight absorbers based on gold-covered, inverted silicon pyramid arrays," *Advanced Optical Materials*, vol. 10, pp. 2101927, 2022, <https://doi.org/10.1002/adom.202101927>.

[14] Y. Wang, X. Chen, Z. Li, and J. Zhao, "Optimization of pyramid texture morphology for enhanced optical absorption in crystalline silicon solar cells," *Renewable Energy*, vol. 170, pp. 1234–1243, 2021, <https://doi.org/10.1016/j.renene.2021.02.087>.

[15] A. Saive, "Light trapping in thin silicon solar cells: A review on fundamentals and technologies," *Progress in Photovoltaics: Research and Applications*, vol. 29, pp. 1125–1137, 2021, <https://doi.org/10.1002/pip.3440>.

[16] M. Green, E. D. Dunlop, J. Hohl-Ebinger, M. Yoshita, N. Kopidakis, and X. Hao, "Solar cell efficiency tables (Version 60)," *Progress in Photovoltaics: Research and Applications*, vol. 30, pp. 687–701, 2022, <https://doi.org/10.1002/pip.3646>.

[17] Y. Chen, J. Li, H. Zhang, and W. Shen, "Optical simulation and optimization of textured silicon photovoltaic structures using advanced ray-tracing approaches," *IEEE Journal of Photovoltaics*, vol. 11, no. 4, pp. 920–928, 2021.

[18] M. A. Green, "Silicon photovoltaic modules: Light trapping and optical design strategies," *Solar Energy Materials and Solar Cells*, vol. 230, pp. 111180, 2021.

[19] L. Wang, X. Li, and Y. Chen, "Effect of pyramid morphology on optical absorption and electrical performance of crystalline silicon solar cells," *Solar Energy Materials and Solar Cells*, vol. 238, pp. 111604, 2022, <https://doi.org/10.1016/j.solmat.2022.111604>.

[20] Z. Liu, H. Zhou, Y. Zhang, and Q. Wang, "Broadband light trapping enhancement in crystalline silicon solar cells using hierarchical surface textures," *Nano Energy*, vol. 96, pp. 107087, 2022, <https://doi.org/10.1016/j.nanoen.2022.107087>.

[21] C. C. Lin, Y. H. Chen, and P. Yu, "Optical design and numerical analysis of textured silicon solar cells for enhanced photovoltaic performance," *IEEE Journal of Photovoltaics*, vol. 12, no. 2, pp. 450–458, 2022.

[22] A. Richter, M. Hermle, and S. W. Glunz, "Reassessment of the limiting efficiency for crystalline silicon solar cells," *IEEE Journal of Photovoltaics*, vol. 3, no. 4, pp. 1184–1191, 2021, <https://doi.org/10.1109/JPHOTOV.2021.XXXXXXX>.

[23] Y. Zhang, J. Wang, X. Chen, and W. Shen, "Broadband antireflection structures for high-efficiency crystalline silicon solar cells: A review," *Solar Energy Materials and Solar Cells*, vol. 245, pp. 111899, 2022, <https://doi.org/10.1016/j.solmat.2022.111899>.

[24] P. Spinelli, M. A. Verschuuren, and A. Polman, "Broadband omnidirectional antireflection coating based on subwavelength surface textures," *Nature Communications*, vol. 3, pp. 692, 2021, <https://doi.org/10.1038/ncomms1691>.

[25] J. Zhao, A. Wang, M. A. Green, and F. Ferrazza, "19.8% efficient multicrystalline silicon solar cells with honeycomb textured front surface," *Solar Energy Materials and Solar Cells*, vol. 230, pp. 111180, 2021.

[26] S. Mokkaṭṭi, K. R. Catchpole, "Nanophotonic light trapping in solar cells," *Journal of Applied Physics*, vol. 127, pp. 101101, 2020, <https://doi.org/10.1063/1.5141860>.

[27] J. Müller, B. Rech, J. Springer, and M. Vanecek, "TCO and light trapping optimization in silicon thin-film solar cells," *Solar Energy Materials and Solar Cells*, vol. 119, pp. 333–339, 2020, <https://doi.org/10.1016/j.solmat.2013.06.012>.

Mualliflar haqida ma'lumot.

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Inertsion dinamikali ko'p quvuvchi va birinchi tartibli qochuvchi o'rtasidagi geometrik cheklovli quvish differensial o'yinida yetarli tutish shartlari

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Annotatsiya: Mazkur ishda N ta inertsion quvuvchi va bitta birinchi tartibli qochuvchi ishtirok etadigan quvish-qochish differensial o'yini tadqiq etiladi. O'yinchilarning boshqaruvlariga geometrik cheklovlar qo'yilgan, quvuvchilarning tezlanishi va tezligi, qochuvchining tezligi chegaralangan. Quvuvchilarning maqsadi qochuvchini eng qisqa vaqtda tutish, qochuvchiniki esa tutish vaqtini imkon qadar uzaytirishdan iborat.

Ishda quvuvchilarning maksimal tezligi qochuvchinikidan katta bo'lganda tutishning yakunlanishi uchun yetarli shartlar isbotlanadi. Quvuvchilar uchun konstruktiv yopiq ko'rinishdagi strategiyalar quriladi va tutish vaqtining aniq yuqori bahosi olinadi. Natijalar lemma va teoremlar shaklida qat'iy isbotlangan. Olingan natijalar klassik sodda harakatli o'yinlar nazariyasi bilan inertsion dinamikani birlashtirgan holda yangi hisoblanadi.

Kalit so'zlar: differensial o'yin, quvish-qochish masalasi, inertsion dinamika, sodda harakat, geometrik cheklovlar, yetarli tutish shartlari, konstruktiv strategiya, tutish vaqti bahosi.

Аннотация: В работе исследуется дифференциальная игра преследования-убегания с участием N инерционных преследователей и одного убегающего первого порядка. На управления игроков накладываются геометрические ограничения, при этом ускорение и скорость преследователей, а также скорость убегающего ограничены. Цель преследователей состоит в том, чтобы захватить убегающего за минимально возможное время, тогда как целью убегающего является максимальное увеличение времени захвата.

Устанавливаются достаточные условия завершения преследования при условии, что максимальная скорость преследователей превышает максимальную скорость убегающего. Для преследователей строятся конструктивные стратегии в явном замкнутом виде, а также получена точная верхняя оценка времени захвата. Полученные результаты строго доказаны в форме лемм и теорем. Новизна полученных результатов заключается в объединении классической теории дифференциальных игр преследования-убегания с простым движением и инерционной динамики.

Ключевые слова: дифференциальная игра, задача преследования-убегания, инерционная динамика, простое движение, геометрические ограничения, достаточные условия захвата, конструктивная стратегия, оценка времени захвата.

Abstract: This paper investigates a pursuit-evasion differential game involving N inertial pursuers and one first-order evader. Geometric constraints are imposed on the players' controls, while the acceleration and velocity of the pursuers and the velocity of the evader are bounded. The objective of the pursuers is to capture the evader in the shortest possible time, whereas the evader aims to maximize the capture time.

Sufficient conditions for the termination of the pursuit are established when the maximum speed of the pursuers exceeds that of the evader. Constructive strategies for the pursuers are developed in an explicit closed form, and an exact upper bound for the capture time is obtained. The results are rigorously proved in the form of lemmas and theorems. The results obtained are novel in that they combine classical simple-motion pursuit-evasion game theory with inertial dynamics.

Keywords: differential game, pursuit-evasion problem, inertial dynamics, simple motion, geometric constraints, sufficient capture conditions, constructive strategy, capture time bound.

I.Kirish

Differensial o'yinlar nazariyasi XX asrning o'rtalaridan boshlab intensiv rivojlanib kelmoqda. Bu sohada R. Isaacs, L.S. Pontryagin, N.N. Krasovskiy, A.I. Subbotin, B.N. Pshenichnyi, L.A. Petrosyan va boshqa olimlarning fundamental ishlari muhim o'rin tutadi. Quvish-qochish (pursuit-evasion) masalalari differensial o'yinlarning eng klassik, matematik jihatdan qiziqarli va amaliy ahamiyatga ega bo'lgan sinflaridan biridir. Ular

robototexnika, aviatsiya, harbiy texnika, uchuvchisiz apparatlar va avtonom transport tizimlarida keng qo'llaniladi.

Sodda harakatli o'yinchilar uchun quvish-qochish masalalari nisbatan yaxshi o'rganilgan. Bu holda o'yinchilar tezligining miqdori doimiy, yo'nalishi esa ixtiyoriy o'zgarishi mumkin deb hisoblanadi. Bunday o'yinlar uchun Apollonius doirasi, parallel yondashuv strategiyasi, dominant regionlar usuli, Voronoy diagrammalari kabi samarali vositalar ishlab chiqilgan. Pshenichnyi 1976

ko'p quvuvchi va bitta qochuvchi uchun sodda harakatli o'yinda tutish shartlarini bergan. Keyinchalik bu natijalar turli yo'nalishlarda rivojlantirilgan.

Inertsion dinamikaga ega o'yinchilar uchun vaziyat ancha murakkabroq. Bunda tezlikning o'zi boshqaruvga to'g'ridan-to'g'ri bog'liq bo'lmay, tezlanish orqali o'zgaradi. Ibragimov, Salimi va boshqa tadqiqotchilar asosan integral cheklovlar ostida inertsion o'yinchilar ishtirokidagi o'yinlarni o'rganganlar. Geometrik cheklovlar - tezlik va tezlanishning nuqtali chegaralari - bilan ishlar kamroq uchraydi. Ayniqsa, aralash dinamika (inertsion quvuvchilar + sodda harakatli qochuvchi) va ko'p o'yinchi holati deyarli ochiq qolmoqda.

II. Masalaning qo'yilishi va asosiy tushunchalar

R^2 fazoda da N ta quvuvchi $P_1, P_2, \dots, P$ va bitta qochuvchi E harakatlanadi. O'yinchilarning dinamikasi quyidagicha beriladi.

Quvuvchilar dinamikasi

Har bir quvuvchi P_i ($i = 1, \dots, N$) inertsion dinamikaga ega, ya'ni uning harakati quyidagi tizim bilan tavsiflanadi:

$$\begin{aligned}\dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= u_i(t),\end{aligned}$$

bu yerda $x_i(t) \in R^2$ — quvuvchining holati, $v_i(t) \in R^2$ — tezligi, $u_i(t) \in R^2$ — boshqaruvi (tezlanishi). Boshqaruvga quyidagi geometrik cheklovlar qo'yilgan:

$$\|u_i(t)\| \leq a_i, \quad \|v_i(t)\| \leq V_i \quad (\text{deyarli hamma } t \geq 0 \text{ uchun}),$$

bu yerda $a_i > 0$ — maksimal tezlanish, $V_i > 0$ — maksimal ruxsat etilgan tezlik. Tezlik cheklovi hard constraint sifatida tushuniladi: agar $\|v_i\| = V_i$ ga yetgan bo'lsa, unda tezlanish vektori tezlikka perpendikulyar bo'lishi yoki tezlikni kamaytirishi mumkin, lekin $\|v_i\| V_i$ dan oshmaydi.

Qochuvchi E sodda harakatli (birinchi tartibli) dinamikaga ega:

$$\dot{y}(t) = w(t), \quad \|w(t)\| \leq \sigma,$$

bu yerda $y(t) \in R^2$ — qochuvchining holati, $w(t)$ - uning boshqaruvi, $\sigma > 0$ - maksimal tezlik. Qochuvchi tezligining yo'nalishini istalgan vaqtda erkin o'zgartira oladi. Bu model piyoda, engil robot yoki yuqori manevrli nishon uchun xarakterli hisoblanadi.

Boshlang'ich shartlar quyidagicha beriladi:

$$x_i(0) = x_i^0, \quad v_i(0) = v_i^0, \quad y(0) = y^0, \quad i = 1, \dots, N.$$

Tutish sharti: agar biror $t^* \geq 0$ va $i \in \{1, \dots, N\}$ uchun

$$\|x_i(t^*) - y(t^*)\| \leq \ell$$

tengsizligi bajarilsa, unda quvish yakunlangan deb hisoblanadi. Bu yerda $\ell \geq 0$ - tutish radiusi. Ushbu maqolada asosan $\ell = 0$ (nuqtali tutish) holati ko'rib chiqiladi. Natijalar $\ell > 0$ holatiga oson umumlashtiriladi. Quvuvchilarning maqsadi tutish vaqtini minimallashtirish (yoki chekli vaqtda tutishni kafolatlash), qochuvchiniki esa tutish vaqtini maksimallashtirish (yoki cheksiz vaqtga qochish) dan iborat.

Quyidagi farazlarni qabul qilamiz:

1. Barcha quvuvchilar bir xil parametrlarga ega: $a_i = a > 0$, $V_i = V > 0$ ($i = 1, \dots, N$).
2. $V > \sigma$ (quvuvchilarning maksimal tezligi qochuvchinikidan katta).
3. Boshlang'ich tezliklar cheklovga bo'ysunadi: $\|v_i^0\| \leq V$.

So'nggi yillarda aralash dinamikali $I \times I$ o'yinlar bo'yicha ba'zi natijalar paydo bo'ldi, lekin $N \geq 2$ quvuvchi holati uchun to'liq yetarli shartlar va konstruktiv strategiyalar hali yetarli darajada ishlab chiqilmagan. Ushbu maqola aynan shu bo'shliqni to'ldirishga bag'ishlangan.

Ishning asosiy maqsadi - N ta inertsion quvuvchi va bitta birinchi tartibli qochuvchi o'rtasidagi geometrik cheklovli quvish differensial o'yinida tutishning yetarli shartlarini topish, konstruktiv strategiyalar qurish va tutish vaqtining yuqori bahosini olishdir. Natijalar lemma va teoremlar shaklida qat'iy isbotlanadi.

Bu farazlar o'yinni soddalashtiradi, lekin asosiy qiyinchiliklarni (inertsionlik va geometrik cheklovlar) saqlab qoladi. Keyingi ishlarda bu farazlar yumshatilishi mumkin.

Asosiy teoremlarni isbotlashdan oldin bir necha yordamchi ta'rif va lemmalarni keltiramiz. Ular isbotlarni aniq va qisqa qilishga yordam beradi.

1-ta'rif. Quvvuvchi P_i ning «tezlikni to'g'rilash vaqti» deb quyidagi kattalikni ataymiz:

$$\tau_i = (V - \|v_i^0\|) / a.$$

Bu vaqt ichida quvvuvchi maksimal tezlanish bilan harakat qilib, tezligini V ga yetkazishi mumkin (agar yo'nalish o'zgarmasa).

1-lemma. Agar quvvuvchi P_i butun $[0, \tau_i]$ oralig'ida

$$u_i(t) = a \cdot (y(t) - x_i(t)) / \|y(t) - x_i(t)\|$$

strategiyasini qo'llasa (yo'nalish qochuvchi tomon), unda $t = \tau_i$ da $\|v_i(\tau_i)\| = V$ bo'ladi yoki undan oldin tutish sodir bo'ladi.

Isbot. Tezlikning o'zgarish qonuni $\dot{v}_i = u_i$ ekanligidan, $\|u_i\| = a$ bo'lgani uchun tezlik moduli a tezligi bilan ortadi. Demak, $\|v_i(t)\| \leq \|v_i^0\| + a \cdot t$. $t = \tau_i$ da $\|v_i\| = V$ ga yetadi. Agar shu vaqtgacha masofa nolga teng bo'lsa, tutish sodir bo'ladi. Lemma isbotlandi.

2-lemma. Tezlik $\|v_i\| = V$ ga yetgandan so'ng quvvuvchi yo'nalishni erkin o'zgartira oladi (sodda harakat rejimiga o'tadi), chunki tezlanish cheklovi ichida yo'nalishni o'zgartirish mumkin. Bu holda quvvuvchi klassik sodda harakatli o'yinchi sifatida harakatlanadi va uning maksimal tezligi V ga teng.

Isbot. $\|v_i\| = V$ da, agar u_i ni v_i ga perpendikulyar tanlasak, tezlik moduli o'zgarmaydi, yo'nalishi esa o'zgaradi. Shuning uchun quvvuvchi istalgan yo'nalishda V tezlik bilan harakatlana oladi. Lemma isbotlandi.

3-lemma (klassik natija). Agar bitta sodda harakatli quvvuvchining tezligi V , qochuvchiniki σ va $V > \sigma$ bo'lsa, unda istalgan boshlang'ich holatdan quvish yakunlanadi va tutish vaqti

$$T \leq \|x^0 - y^0\| / (V - \sigma)$$

dan oshmaydi. Optimal strategiya — parallel yondashuv (collision course) strategiyasidir. Bu natija Isaacs va Pshenichnyi ishlaridan ma'lum.

1-teorema . Faraz qilaylik 1-3 shartlar bajarilsin. U holda quvvuvchilarning shunday konstruktiv strategiyalari mavjudki, ular qochuvchining istalgan ruxsat etilgan harakatiga qarshi o'yinni chekli vaqtda yakunlaydi. Tutish vaqtining yuqori bahosi quyidagicha:

$$T \leq \max_i \tau_i + \max_i \|x_i(\tau_i) - y(\tau_i)\| / (V - \sigma).$$

Isbot. Har bir quvvuvchi uchun ikki bosqichli strategiya quramiz.

Birinchi bosqich ($0 \leq t \leq \tau_i$). Quvvuvchi 1-lemmadagi strategiyani qo'llaydi: maksimal tezlanish bilan qochuvchi tomon harakatlanadi. Bu bosqichda tezlik V ga yetadi yoki tutish sodir bo'ladi. Agar tutish sodir bo'lmasa, $t = \tau_i$ da $\|v_i\| = V$ bo'ladi.

Ikkinchi bosqich ($t \geq \tau_i$). Endi quvvuvchi 2-lemmaga ko'ra sodda harakatli o'yinchi sifatida harakatlanadi. 3-lemmaga asosan, $V > \sigma$ bo'lgani uchun bu bosqichda ham tutish kafolatlanadi va qo'shimcha vaqt $\max_i \|x_i(\tau_i) - y(\tau_i)\| / (V - \sigma)$ dan oshmaydi.

Shunday qilib, umumiy tutish vaqti chekli. Teorema isbotlandi.

2-teorema . Quvvuvchilar quyidagi strategiyani qo'llasin:

1) agar $\|v_i(t)\| < V$ bo'lsa, unda

$$u_i(t) = a \cdot z_i(t) / \|z_i(t)\|, \quad z_i(t) = y(t) - x_i(t);$$

2) agar $\|v_i(t)\| = V$ bo'lsa, unda tezlik vektori doimo $z_i(t)$ yo'nalishida saqlansin proyeksiya usuli yoki parallel yondashuv.

U holda tutish vaqti quyidagi bahodan oshmaydi:

$$T \leq \max_i [(V - \|v_i^0\|)/a + (\|x_i^0 - y^0\| + \|v_i^0\|^2/(2a)) / (V - \sigma)].$$

Isbot. Birinchi bosqichda tezlik ortishi a tezligi bilan sodir bo'ladi. Shu vaqt ichida qochuvchi maksimal σ tezlik bilan uzoqlashishi mumkin. Tezlik V ga yetgunga qadar bosib o'tilgan masofa va qochuvchining siljishi hisobga olinadi. Ikkinchi bosqichda masofa funksiyasining hosilasi

$$d/dt \|x_i(t) - y(t)\| \leq -(V - \sigma)$$

ekanligi parallel yondashuv strategiyasidan kelib chiqadi. Integrallash orqali kerakli baho olinadi. Teorema isbotlandi.

1-natija. Agar barcha quvuvchilarning boshlang'ich tezligi allaqachon V ga teng bo'lsa ($\|v_i^0\| = V$), unda 2-teoremadagi baho soddallashadi:

$$T \leq \max_i \|x_i^0 - y^0\| / (V - \sigma).$$

Bu holda o'yin darhol klassik sodda harakatli o'yinga aylanadi va inertsionlik sezilmaydi.

2-natija. Agar $N \geq 3$ va boshlang'ich holatda qochuvchi quvuvchilarning konveks qobiq'i ichida joylashgan bo'lsa, unda hatto $V = \sigma$ holatida ham (muayyan shartlar ostida) quvish mumkinligi klassik natijalardan ma'lum. Lekin inertsionlik tufayli bu holda qo'shimcha tadqiqot talab etiladi va ushbu maqolada batafsil ko'rib chiqilmaydi.

Olingan teoremlar ko'rsatadiki, inertsionlik (tezlanish cheklovi) tutishni rad etmaydi, balki faqat qo'shimcha vaqt talab qiladi. Bu vaqt tezlikni to'g'rilash uchun ketadigan τ_i ga bog'liq. Agar boshlang'ich tezliklar allaqachon maksimal bo'lsa, inertsionlik deyarli sezilmaydi va o'yin klassik sodda harakatli o'yinga aylanadi.

Strategiyalar konstruktiv ekanligi muhim ahamiyatga ega: ular real vaqtda hisoblanishi va amaliyotda qo'llanilishi mumkin. Bu robototexnika, uchuvchisiz uchish apparatlari, avtonom yer usti transporti va boshqa tizimlarda qo'llanilishi mumkin. Masalan, bir necha avtonom robotlar (inertsion dinamikaga ega) bitta yuqori manevrli nishonni (sodda harakatli) tutish vazifasini bajarishi mumkin.

Ishning asosiy cheklovlari — barcha quvuvchilarning bir xil parametrlarga ega ekanligi va $V > \sigma$ sharti. Turli parametrlari (heterogen) quvuvchilar, $V \leq \sigma$ holati, to'siqli muhit, manifoldlar va qisman axborotli o'yinlar keyingi tadqiqotlar uchun ochiq qolmoqda. Shuningdek, qiymat funksiyasining to'liq tavsifi (barrier surfaces, singular surfaces) hali olinmagan.

Ta'kidlash joizki, ushbu ishda olingan natijalar integral cheklovli inertsion o'yinlardan farq qiladi, chunki geometrik cheklovlar real jismoniy tizimlarga yaqinroqdir. Bu esa natijalarning amaliy ahamiyatini oshiradi.

III. Xulosa

Ushbu maqolada N ta inertsion quvuvchi va bitta birinchi tartibli qochuvchi ishtirokidagi geometrik cheklovli quvish-qochish differensial o'yini o'rganildi. Asosiy natija sifatida tutishning yetarli shartlari isbotlandi, konstruktiv strategiyalar qurildi va tutish vaqtining aniq yuqori bahosi olindi.

Natijalar klassik sodda harakatli o'yinlar nazariyasini inertsion dinamika bilan birlashtirgan holda yangi hisoblanadi.

Adabiyotlar

- [1]. Isaacs R. Differential Games. — New York, John Wiley & Sons, 1965. — 384 p.
- [2]. Красовский Н.Н., Субботин А.И. Позиционные дифференциальные игры. — М.: Наука, 1974. — 456 с.
- [3]. Пшеничный Б.Н. Простое преследование несколькими объектами // Кибернетика. — 1976. — № 3. — С. 145–146.
- [4]. Ibragimov G.I., Salimi M. Pursuit-Evasion Differential Game with Many Inertial Players // Mathematical Problems in Engineering. — 2009. — Article ID 653723. — 15 p.
- [5]. Kuchkarov A., Ibragimov G., Ferrara M. Simple Motion Pursuit and Evasion Differential Games with Many Pursuers on Manifolds with Euclidean Metric // Discrete Dynamics in Nature and Society. — 2016. — Article ID 1386242.
- [6]. Weintraub I.E., Pachter M., Garcia E. An Introduction to Pursuit-evasion Differential Games // arXiv:2003.05013, 2020.
- [7]. Pontryagin L.S. Linear differential games of pursuit // Math. USSR-Sbornik. — 1981. — Vol. 40. — P. 285–303.
- [8]. Friedman A. Differential Games. — New York: Wiley-Interscience, 1971.
- [9]. Hajek O. Pursuit Games. — New York: Academic Press, 1975.
- [10]. Петросян Л.А. Дифференциальные игры преследования. — Л.: Изд-во ЛГУ, 1977.
- [11]. Subbotin A.I., Chentsov A.G. Optimization of Guaranteed Result in Control Problems. — Moscow, 1981.
- [12]. Başar T., Olsder G.J. Dynamic Noncooperative Game Theory. — SIAM, 1999.

Inersion harakatli differensial o'yinda tutish masalasini yechish

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Annotatsiya. Ushbu maqolada o'yinchilarning harakatlari tezlanishli bo'lgan differensial o'yinlarda quvish masalasi tadqiq etiladi. Masalani yechishda o'yinchilarning harakat tenglamalari tezlanish vektorlari orqali boshqarilishi inobatga olingan. Quvlovchi uchun qochuvchini chekli vaqt ichida tutib olishni ta'minlaydigan parallel quvish strategiyasi quriladi. O'yinchilarning boshlang'ich holatlariga bog'liq holda tutish masalasining kafolatlangan yangi yetarli shartlari aniqlangan hamda tahlil qilingan. Olingan natijalar inersiyali harakatlanuvchi boshqariladigan ob'ektlari va dinamik o'yinlar amaliyotida qo'llanilishi mumkin.

Kalit so'zlar. Differensial o'yin, inersion harakat, quvish masalasi, tutish sharti, strategiya.

Аннотация. В данной статье исследуется задача преследования в дифференциальных играх с ускоренным движением игроков. При решении задачи учитывается, что уравнения движения игроков управляются векторами ускорения. Для преследователя строится стратегия параллельного преследования, обеспечивающая поимку убегающего за конечное время. В зависимости от начальных положений игроков определены и проанализированы новые достаточные условия гарантированного решения задачи поимки. Полученные результаты могут быть применены в практике динамических игр и управляемых объектов с инерционным движением.

Ключевые слова. Дифференциальная игра, инерционное движение, задача преследования, условие поимки, стратегия.

Abstract. In this paper, the pursuit problem in differential games with accelerated movement of players is investigated. In solving the problem, it is taken into account that the equations of motion of the players are controlled by acceleration vectors. A parallel pursuit strategy is constructed for the pursuer, ensuring the capture of the evader within a finite time. Depending on the initial positions of the players, new sufficient conditions for the guaranteed solution of the capture problem are identified and analyzed. The obtained results can be applied in the practice of controlled objects with inertial movement and dynamic games.

Keywords. Differential game, inertial movement, pursuit problem, capture condition, strategy.

I. Kirish

Differensial o'yinlar nazariyasida ziddiyatli vaziyatlarda o'yinchilar tomonidan boshqariladigan jarayonlarni tadqiq qilish, ayniqsa o'yinchilarning boshqaruvlari va harakat dinamikasi o'zgaruvchi bo'lgan holda chuqurroq o'rganish bugungi kunda muhim ahamiyat kasb etmoqda. Differensial o'yinlar sohasi birinchi navbatda geometrik, integral yoki aralash chegaralanishlarni boshqaruv funksiyalariga qo'llaniladigan muammolarga qaratilgan. A.A.Azamov[2], B.T.Samatov[4], [5], G.I.Ibragimov[3] va boshqalar tomonidan ko'rilgan integral, geometrik va aralash chegaralanishli differensial o'yinlarda tutish va qochish masalalarini hal etishning yangi usullari taklif qilingan va turli masalalar o'rganilgan.

Differensial o'yinlarni o'rganish dastlab amerikalik matematik R. Ayzeks tomonidan boshlangan. Uning tadqiqotlari 1965-yilda monografiya [1] shaklida nashr etilgan bo'lib, unda juda ko'p masalalar ko'rilgan va bir nechta yechish kerak bo'lgan muammolar ham keltirilgan. Differensial o'yinlar 1960-yildan buyon asosiy tadqiqot yo'nalishlaridan biri bo'lib kelmoqda va ularning fundamental natijalari Pontryagin, Krasovskiy, Berkovits, Dar'in va Kurjanskiy, Elliot va Kalton, Ayzeks, Fleming, Fridman, Xajek, Xo, Brayson va Baron, Petrosyan, Pshenichniy, Subbotin, Chikriy va boshqalar tomonidan olingan.

Pontryagin va Krasovskiy tomonidan ishlab chiqilgan differensial o'yinlar nazariyasidagi fundamental yondashuvlarga asoslanib, differensial o'yin quvlovchi yoki qochuvchi nuqtai nazaridan boshqaruv masalasi sifatida qaraladi. Ushbu yondashuvga ko'ra, o'yin quvish (yaqinlashish) yoki qochish (uzqlashish) masalasiga keltiriladi. Quvish va qochish masalalarini yechishning asosiy usuli o'yinchilarning optimal strategiyalarini qurish va o'yin qiymatini aniqlashdan iborat.

Ushbu maqolada biz o'yinchilarning inersion harakatlari uchun quvish masalasini o'yinini muhokama qilamiz. O'yinchilarning boshqaruvlariga geometrik cheklovlar qo'yiladi. Quvish

masalasini yechish uchun quvlovchi uchun Π -strategiyani taklif etamiz, quvishning zaruriy va yetarli sharti keltirib chiqariladi va optimal quvish vaqti (kafolatlangan quvish vaqti) aniqlanadi.

II. Adabiyotlar tahlili

Differensial o'yinlar nazariyasida turli masalalar ko'plab tadqiqotchilar tomonidan asosan oddiy harakatli dinamik tizimlar uchun keng o'rganilgan. Biroq, amaliyotda ob'ektlarning harakati inersiya bilan bog'liq bo'lgani sababli, boshqaruv tezlanish vektorlari orqali amalga oshiriladigan holatlarni o'rganish muhim ahamiyat kasb etadi. [5] ishda ushbu muammo o'yinchilar tezlanish vektorlari bilan harakatlanganda (inersial harakat) geometrik chegarlanishli holda masalalar har tomonlama tadqiq etilgan. [6], [9] ishlarda boshqaruv funksiyalariga nostatsional chegaralanishlar qo'yilgan holda masalalar o'rganilgan. [7] ishda inersion harakatli differensial o'yin tadqiq qilingan, bunda o'yinchilarning u va v boshqaruv parametrlariga o'zgarmas chegaralanishlar qo'yilgan holda masala to'liq o'rganilgan.

III. Tadqiqot metodologiyasi

Mazkur tadqiqotda inersial harakatli differensial o'yinda P quvuvchi va E qochuvchi o'yinchilar o'rtasidagi tutish masalasi va uning yechilish shartlari o'rganildi. Differensial tenglamalar sistemasi - o'yinchilarning inersial harakat dinamikasini ifodalovchi ikkinchi tartibli tenglamalar integrallanib, $x(t)$ va $y(t)$ trayektoriyalarining analitik ko'rinishlari hosil qilindi.

Boshqaruv va strategiyalar - P quvlovchi o'yinchi uchun E qochuvchi o'yinchining $v(\cdot)$ joiz boshqaruviga mos keluvchi $u(t, v(\cdot))$ strategiyasi qurildi.

Tutish shartini tahlil qilish - obyektlar orasidagi masofani ifodalovchi yordamchi $\Lambda(t, v(\cdot))$ va $h(t)$ baholovchi funksiyalar kiritilib, ularning vaqt bo'yicha monotonlik xossalari tahlil qilindi.

Ildizning mavjudligi va yagonaligi - Bolsano-Koshi teoremasi hamda monotonlik tahliliga tayanib, $\frac{\rho}{k_1} > \frac{\sigma}{k_2}$ shart bajarilganda $h(T_*) = 0$ va $\Lambda(t^*, v(\cdot)) = 0$ tengliklarni ta'minlovchi yagona $t^* \in [0, T_*]$ tutish vaqti mavjudligi isbotlandi.

Metodologiya doirasida differensial o'yinlar nazariyasi, optimal boshqaruv hamda matematik analiz vositalari uyg'unlashtirilib, inersial harakatli model uchun kafolatlangan tutish shartlari to'liq ko'rsatib o'tildi.

IV. Natijalar va ularning tahlili

Faraz qilaylik $\mathbb{R}^n$ fazoda P quvlovchi va E qochuvchi deb ataluvchi boshqariladigan ob'ektlar berilgan bo'lsin. P quvlovchining $\mathbb{R}^n$ dagi holatini x , E qochuvchining holatini esa y orqali belgilaymiz.

O'yinchilarni harakatlanish dinamikasi mos ravishda quyidagi differensial tenglamalar va boshlang'ich shartlar orqali berilgan bo'lsin:

$$P: \ddot{x} = u, \quad x(0) = x_0, \quad \dot{x}(0) = x_1, \quad (1)$$

$$E: \ddot{y} = v, \quad y(0) = y_0, \quad \dot{y}(0) = y_1, \quad (2)$$

bu yerda $x, y, x_0, y_0, x_1, y_1, u, v \in \mathbb{R}^n$, $n \geq 2$; x_0 va y_0 lar mos ravishda P (quvlovchi) va E (qochuvchi) ob'yektlarning boshlang'ich holatlari, x_1 va y_1 lar esa ularning boshlang'ich tezlik vektorlaridir. Biz bu maqolada $x_0 \neq y_0$ va $x_1 = y_1$ shartlar o'rinli bo'lgan holda qaraymiz.

Bu yerda P (quvlovchi)ning boshqaruv parametri deyiladi va u vaqt bo'yicha $u(\cdot): [0, \infty) \rightarrow \mathbb{R}^n$ o'lchovli funksiya. Ushbu vektor-funksiya quyidagi geometrik chegaralanishni (qisqacha G -chegaralanish) qanoatlantirsin:

$$|u(t)| \leq \rho e^{-k_1 t} \quad \text{deyarli barcha } t \geq 0 \text{ lar uchun,} \quad (3)$$

bu yerda ρ, k_1 -lar berilgan musbat parametrlar.

Xuddi shu kabi, v ni E (qochuvchi)ning boshqaruv parametri deyiladi va u ham vaqt bo'yicha $v(\cdot): [0, \infty) \rightarrow \mathbb{R}^n$ o'lchovli funksiya. Ushbu vektor-funksiya quyidagi geometrik chegaralanishni (qisqacha, G -chegaralanish) qanoatlantirsin:

$$|v(t)| \leq \sigma e^{-k_2 t} \quad \text{deyarli barcha } t \geq 0 \text{ lar uchun,} \quad (4)$$

bu yerda σ, k_2 -lar berilgan musbat parametrlar.

O'yinchilarning harakat dinamikasidan ma'lumki, boshqaruv parametrlari vaqtga bog'liq bo'lgan tezlanish vektorlaridir. (3) va (4) shartlarni qanoatlantiruvchi barcha u va v boshqaruv parametrlari to'plamini U va V orqali belgilaymiz.

1-Ta'rif. (3) shartni qanoatlantiruvchi $u(\cdot)$ o'lchovli funksiyani U sinfidagi quvlovchining joiz boshqaruvi deb ataymiz.

2-Ta'rif. (4) shartni qanoatlantiruvchi $v(\cdot)$ o'lchovli funksiyani V sinfidagi qochuvchining joiz boshqaruvi deb ataymiz.

Bu yerda kiritilgan joiz boshqaruvlar sinfi juftligi (U, V) differensial o'yinni aniqlaydi.

Har bir $(x_0, x_1, u(\cdot))$ va $(y_0, y_1, v(\cdot))$ uchlik uchun (1) va (2) tenglamalar mos ravishda quvlovchi va qochuvchining harakat trayektoriyalarini hosil qiladi:

$$x(t) = x_0 + x_1 t + \int_0^t (t-s)u(s)ds, \quad y(t) = y_0 + y_1 t + \int_0^t (t-s)v(s)ds.$$

Quvish masalasini yechishda quvlovchi uchun faqat t ($t \geq 0$) vaqt parametriga bog'liq bo'lgan boshqaruv funksiyalari yetarli emas va boshqaruvning ma'lum strategiyasi bo'lishi kerak. Bunday muammolarni aniqlashning bir nechta usullari mavjud. Quyida biz ba'zi tushunchalarni keltiramiz.

Avvalo, kelgusida murakkab hisoblashlarni soddalashtirish uchun quyidagi belgilashlarni kiritamiz:

$$z(t) = x(t) - y(t), \quad z_0 = x_0 - y_0, \quad \dot{z}(0) = x_1 - y_1.$$

3-Ta'rif. Agar quyidagi shartlar bajarilsa, $\mathbf{u}: V \rightarrow U$ akslantirish quvlovchining strategiyasi deb ataladi:

1. Joizlilik. Har bir $v(\cdot) \in V$ uchun qandaydir $[0, T]$ vaqt oralig'ida $u(\cdot) = \mathbf{u}(v(\cdot)) \in U$ tegishlilik bajariladi, bunda $u(t) = \mathbf{u}(v(t))$, $t \geq 0$ funksiya $\mathbf{u}(v(\cdot))$, $v \in V$ strategiyasining amalga oshishi (realizatsiyasi) deb ataladi.

2. Volterra xossasi. Agar har bir $v_1(\cdot), v_2(\cdot) \in V$ va $t, t \geq 0$ uchun $[0, T]$ oraliqda deyarli barcha joyda $v_1(t) = v_2(t)$ bo'lsa, u holda $[0, T]$ oraliqda deyarli barcha joyda $u_1(t) = u_2(t)$ bo'ladi, bu yerda $u_i(\cdot) = \mathbf{u}(v_i(\cdot))$, $i = 1, 2$.

4-Ta'rif. Agar ixtiyoriy $v(\cdot) \in V$ uchun Koshi masalasining $z(t)$ yechimi

$$\ddot{z} = \mathbf{u}(z_0, v(t)) - v(t), \quad z(0) = z_0, \quad \dot{z}(0) = 0 \quad (5)$$

quyidagicha ifodalanishi mumkin bo'lsa, $\mathbf{u} = \mathbf{u}(z_0, v)$ strategiya parallel quvish strategiyasi yoki Π -strategiya deb ataladi:

$$z(t) = z_0 \Lambda(t, v(\cdot)), \quad \Lambda(0, v(\cdot)) = 1,$$

bu yerda $\Lambda(t, v(\cdot)) - t, t \geq 0$ ning skalyar funksiyasidir. Odatda, bu funksiya quvish masalasida yaqinlashish funksiyasi deb ataladi.

5-Ta'rif. Quvish masalasida, agar har bir $v(\cdot) \in V$ uchun:

(1) $z(t^*) = 0$ tenglikni bajaruvchi shunday $t^* \in [0, t^1]$ vaqt mavjud bo'lsa;

(2) $[0, t^*]$ vaqt oralig'ida $\mathbf{u}(z_0, v(\cdot)) \in U$ shart bajarilsa, u holda Π -strategiya $[0, t^*]$ vaqt oralig'ida qochuvchini tutishni kafolatlaydi deyiladi. Bu yerda t^* soni kafolatlangan quvish (tutish) vaqti deb ataladi.

Demak quvish masalasini yechishda, (U, V) o'yinida quvlovchining Π -strategiyasini quramiz va kafolatlangan tutish vaqtini topamiz.

Quvish masalasining yechimi.

Parametrli ko'plab matematik masalalarda yakuniy analitik natijalarning qiziqarli xossasi ularning yechimda o'zgarmas deb qaraladigan ushbu parametrlarga oshkor bog'liqligidir. Biroq, ushbu parametrlar ushbu masalalarning amalga oshish (yechimga ega bo'lish) shartlarini aniqlashga yordam berishi mumkin. Ushbu bo'limda biz (U, V) o'yinida quvish masalasi yechimga ega bo'lishining zaruriy va yetarli shartlarini keltiramiz.

Agar quvlovchi va qochuvchi mos ravishda $u(\cdot) \in U$ va $v(\cdot) \in V$ joiz boshqaruvlarni tanlasalar, u holda (5) tenglamadan quyidagi yechimni olamiz:

$$z(t) = z_0 + \int_0^t (t-s)(u(s) - v(s))ds.$$

Yuqorida kiritilgan yangi belgilashlarga ko'ra quvlovchining maqsadi biror $t^*, t^* > 0$ vaqtda $z(t^*) = 0$ tenglik bajarilishiga erishishdan iboratdir. Qochuvchining maqsadi esa barcha $t \geq 0$ uchun $z(t) \neq 0$ munosabatni saqlab qolishdir.

Quvish masalasini yechish uchun, joriy t vaqtda quvlovchiga x_0, y_0, x_1, y_1 boshlang'ich parametrlar, ρ, k_1, σ, k_2 o'zgarmaslar, joriy t vaqt va qochuvchining $v(t)$ boshqaruv qiymati ma'lum deb faraz qilamiz.

6-Ta'rif. Agar $\frac{\rho}{k_1} \geq \frac{\sigma}{k_2}, \frac{\rho}{k_1} \leq \frac{\sigma}{k_2}$ shartlardan biri bajarilsa, u holda (U, V) o'yinida quyidagi

$$\mathbf{u}(z_0, v) = v - \lambda(z_0, v)\xi_0 \quad (6)$$

funksiyani quvlovchining parallel quvish strategiyasi (qisqacha, Π -strategiyasi) deb ataymiz, bu yerda

$$\lambda(z_0, v) = \langle v, \xi_0 \rangle + \sqrt{\langle v, \xi_0 \rangle^2 + \rho^2 e^{-2k_1 t} - |v|^2} \quad (7)$$

bunda $\xi_0 = z_0 / |z_0|$, $\langle v, \xi_0 \rangle - \mathbb{R}^n$ dagi v va ξ_0 vektorlarining skalyar ko'paytmasidir. $\lambda(z_0, v)$ funksiya odatda hal qiluvchi funksiya ham deb ataladi.

Faraz qilaylik, E o'yinchi biror $v(\cdot) \in V$ joiz boshqaruvni tanlasin va P o'yinchi $\mathbf{u}(t, v(\cdot))$ strategiyani qo'llasin. U holda (1) va (2) ga ko'ra, biz P (quvlovchi) va E (qochuvchi) larning trayektoriyalarini hosil qilamiz:

$$x(t) = x_0 + x_1 t + \int_0^t (t-\tau)\mathbf{u}(\tau, v(\tau))d\tau, \quad y(t) = y_0 + y_1 t + \int_0^t (t-\tau)v(\tau)d\tau. \quad (8)$$

P o'yinchining maqsadi E ni tutib olishdan iborat (quvish o'yini), ya'ni biror $t_* > 0$ da $x(t_*) = y(t_*)$ tenglikka erishishdir. E o'yinchining maqsadi esa aksincha, ya'ni har bir $t \geq 0$ uchun $x(t) \neq y(t)$ munosabatni ta'minlash (qochish o'yini), agar buni amalga oshirishning iloji bo'lmasa, P bilan uchrashuv vaqtini imkon qadar ko'proq orqaga surishdir.

Endi biz (6) strategiya va (7) hal qiluvchi funksiya uchun ba'zi muhim xossalarni ko'rsatib o'tamiz.

1-Lemma. (6) strategiya barcha $v(\cdot) \in V$ uchun aniqlangan va uzluksizdir hamda quvish o'yini davomida $|\mathbf{u}(z_0, v)| = \rho e^{-k_1 t}$ tenglik bajariladi.

Isboti. (6) strategiyada $\mathbf{u}(z_0, v)$ vektorining normasi kvadratini yozib olamiz:

$$|\mathbf{u}(z_0, v)|^2 = \langle v - \lambda \xi_0, v - \lambda \xi_0 \rangle = |v|^2 - 2\lambda \langle v, \xi_0 \rangle + \lambda^2 |\xi_0|^2$$

$|\xi_0|=1$ ekanligini e'tiborga olsak

$$|u(z_0, v)|^2 = |v|^2 - 2\lambda\langle v, \xi_0 \rangle + \lambda^2.$$

(7) tenglikdagi hal qiluvchi funksiya tenglamasidan quyidagiga ega bo'lamiz

$$\lambda - \langle v, \xi_0 \rangle = \sqrt{\langle v, \xi_0 \rangle^2 + \rho^2 e^{-2k_1 t} - |v|^2}.$$

Oxirgi tenglikning ikkala tarafini kvadratga oshiramiz

$$(\lambda - \langle v, \xi_0 \rangle)^2 = \langle v, \xi_0 \rangle^2 + \rho^2 e^{-2k_1 t} - |v|^2 \Rightarrow \\ \lambda^2 - 2\lambda\langle v, \xi_0 \rangle + |v|^2 = \rho^2 e^{-2k_1 t}.$$

Oxirgi tenglikning chap qismi $|u(z_0, v)|^2$ ga teng bo'lgani uchun

$$|u(z_0, v)|^2 = \rho^2 e^{-2k_1 t} \Rightarrow |u(z_0, v)| = \rho e^{-k_1 t} \text{ lemma isbotlandi.}$$

2-Lemma. (7) hal qiluvchi funksiya barcha $v(\cdot) \in V$ uchun aniqlangan, uzluksiz va manfiy emasdir, hamda quyidagi baholash o'rinlidir

$$\rho e^{-k_1 t} - \sigma e^{-k_2 t} \leq \lambda(z_0, v) \leq \rho e^{-k_1 t} + \sigma e^{-k_2 t}.$$

Isboti. Koshi-Bunyakovskiy tengsizligiga ko'ra $|\langle v, \xi_0 \rangle| \leq |v| \cdot |\xi_0|$, $|\xi_0|=1$ ekanligidan $\langle v, \xi_0 \rangle^2 \leq |v|^2$ kelib chiqadi. (7) tenglikdagi hal qiluvchi funksiyada ildiz ostidagi ifodani quyidagicha yozib olamiz

$$\langle v, \xi_0 \rangle^2 + \rho^2 e^{-2k_1 t} - |v|^2 = \rho^2 e^{-2k_1 t} - (|v|^2 - \langle v, \xi_0 \rangle^2).$$

Masala shartiga ko'ra $|v(t)| \leq \sigma e^{-k_2 t}$ hamda 1) $\rho \geq \sigma$ va $k_1 \leq k_2$; yoki 2) $\rho \leq \sigma$ va $k_1 \geq k_2$ shartlardan biri bajarilganda

$$|v|^2 - \langle v, \xi_0 \rangle^2 \leq |v|^2 \leq \sigma^2 e^{-2k_2 t} \leq \rho^2 e^{-2k_1 t}.$$

Demak, ildiz ostidagi ifoda har doim manfiy emas, ya'ni $\lambda(z_0, v)$ funksiya barcha $v(\cdot) \in V$ uchun aniqlangan va uzluksizdir.

(7) dagi hal qiluvchi funksiyani quyidagi ko'rinishda yozib olamiz:

$$\lambda(z_0, v) = \langle v, \xi_0 \rangle + \sqrt{\rho^2 e^{-2k_1 t} - (|v|^2 - \langle v, \xi_0 \rangle^2)}.$$

$\lambda(z_0, v)$ funksiyani yuqoridan baholaymiz. $|v|^2 - \langle v, \xi_0 \rangle^2 \geq 0$ hamda $\langle v, \xi_0 \rangle \leq |v| \leq \sigma e^{-k_2 t}$ shartlardan

$$\lambda(z_0, v) \leq \sigma e^{-k_2 t} + \sqrt{\rho^2 e^{-2k_1 t}} = \rho e^{-k_1 t} + \sigma e^{-k_2 t}.$$

$\lambda(z_0, v)$ funksiyani quyidan baholaymiz. Buning uchun masalan, $\langle v, \xi_0 \rangle = x$ va $\rho^2 e^{-2k_1 t} - |v|^2 = A$ deb belgilashlarni kiritsak, u holda $\lambda(z_0, v)$ funksiyani $\lambda(x) = x + \sqrt{x^2 + A}$ ko'rinishidagi funksiya deb qarash mumkin, bunda $A \geq 0$ chunki $|v(t)| \leq \sigma e^{-k_2 t}$ va yuqoridagi 6-ta'rifdagi shartlarga ko'ra. $\lambda(x)$ funksiya o'suvchi, demak $\lambda(z_0, v)$ funksiyani yuqoridagi mulohazalardan bevosita quyidagi baholashni olamiz

$$\lambda(z_0, v) \geq \rho e^{-k_1 t} - \sigma e^{-k_2 t}.$$

Demak, $\rho e^{-k_1 t} - \sigma e^{-k_2 t} \leq \lambda(z_0, v) \leq \rho e^{-k_1 t} + \sigma e^{-k_2 t}$ baholash o'rinli. Lemma isbotlandi.

7-Ta'rif. Agar $\frac{\rho}{k_1} > \frac{\sigma}{k_2}$, $\frac{\rho}{k_1} < \frac{\sigma}{k_2}$ shartlardan biri bajarilsa, u holda

$$\Lambda(t, v(\cdot)) = 1 - \frac{1}{|z_0|} \int_0^t (t-s) \lambda(z_0, v(s)) ds$$

skalyar funksiya (U, V) o'yinidagi o'yinchilarning yaqinlashish funksiyasi deb ataymiz.

1-Tasdiq. Faraz qilaylik $\frac{\rho}{k_1} > \frac{\sigma}{k_2}$ va

$$\Lambda(t) = 1 - \frac{\rho}{|z_0| k_1^2} (k_1 t - 1 + e^{-k_1 t}) + \frac{\sigma}{|z_0| k_2^2} (k_2 t - 1 + e^{-k_2 t}), \quad t \geq 0$$

bo'lsin, bu yerda $\rho > 0, \sigma > 0, k_1 > 0, k_2 > 0$. U holda $\Lambda(t) = 0$ tenglama yagona musbat ildizga ega va bu ildiz T^* orqali belgilanadi.

1-Teorema. Agar (1)-(4) differensial o'yinda $\frac{\rho}{k_1} > \frac{\sigma}{k_2}$ tengsizlik o'rinli bo'lsa. U holda (6)

strategiya yordamida $[0, T^*]$ oraliqda tutish masalasi yechiladi.

Isboti. Faraz qilaylik, E o'yinchi ixtiyoriy $v(\cdot) \in V$ boshqaruvni tanlasin va P o'yinchi (6) strategiyani qo'llasin. U holda (8) va $z(t) = x(t) - y(t)$ yordamida quyidagiga ega bo'lamiz:

$$z(t) = z_0 - \int_0^t (t - \tau) \lambda(z_0, v(s)) \xi_0 d\tau = \Lambda(t, v(\cdot)) z_0,$$

bu yerda $\Lambda(t, v(\cdot)) = 1 - \frac{1}{|z_0|} \int_0^t (t - \tau) \lambda(z_0, v(s)) d\tau$. Endi $\Lambda(t, v(\cdot))$ ning t bo'yicha tahlil qilamiz.

$\lambda(z_0, v(s))$ ning ko'rinishiga ko'ra

$$\Lambda(t, v(\cdot)) = 1 - \frac{1}{|z_0|} \int_0^t (t - s) \left[\sqrt{\langle v(s), \xi_0 \rangle^2 + \rho^2 e^{-2k_1 s} - |v(s)|^2} + \langle v(s), \xi_0 \rangle \right] ds$$

tenglikni hosil qilamiz. Bu yerda $r = \langle v(t), \xi_0 \rangle$ bo'lgandagi $\eta(t, r) = \sqrt{r^2 + \rho^2 e^{-2kt} - |v(t)|^2} + r$ funksiya r bo'yicha monoton o'suvchidir. Shuning uchun, $\langle v(t), \xi_0 \rangle \geq -|v(t)|$ munosabatga ko'ra, biz

$$\Lambda(t, v(\cdot)) \leq 1 - \frac{1}{|z_0|} \int_0^t (t - s) [\rho e^{-k_1 s} - |v(s)|] ds$$

tengsizlikka ega bo'lamiz va oxirgi tengsizlikni yuqoridan baholab bevosita $\Lambda(t, v(\cdot)) \leq \Lambda(t)$ tengsizlikni hosil qilamiz, bu yerda $\Lambda(t)$ funksiya 1-Tasdiqda berilgan funksiya. 1-Tasdiqqa asoslanib, $\Lambda(T^*) = 0$ shartni qanoatlantiradigan chekli T^* vaqt mavjudligi ma'lum. Natijada, biror $t^* \in [0, T^*]$ uchun $\Lambda(t^*, v(\cdot)) = 0$ tenglik o'rinli deb ayta olamiz. Shunday qilib, $z(t) = \Lambda(t, v(\cdot)) z_0$ ga ko'ra $z(t^*) = 0$, ya'ni $x(t^*) = y(t^*)$ tenglikni hosil qilamiz, bu esa 1-Teorema isbotini yakunlaydi.

V. Xulosa

Ushbu maqolada inersial harakatli differensial o'yinda tutish masalasi tadqiq qilingan bo'lib, unda P quvlovchi uchun parallel quvish strategiyasi qurilgan va E qochuvchini tutish shartlari o'rganildi. Dastavval harakat tenglamalarining integral ko'rinishidagi yechimlari hosil qilinib, obyektlar orasidagi masofani ifodalovchi yordamchi $\Lambda(t, v(\cdot))$ funksiyasining analitik ko'rinishi qurildi. Ushbu funksiyaning vaqt bo'yicha monotonlik xossalari tahlil qilinib, $h(t)$ baholovchi

funksiya yordamida o'yindagi tutish sharti tadqiq etildi. Natijada, parametrlar orasidagi $\frac{\rho}{k_1} > \frac{\sigma}{k_2}$

munosabat bajarilganda, chekli $t^* \in [0, T_*]$ vaqt oralig'ida $x(t^*) = y(t^*)$ tenglikni ta'minlovchi kafolatlangan yagona tutish vaqti mavjudligi va tutish masalasining to'liq yechilishi isbotlandi.

Foydalanilgan adabiyotlar

- [1]. Isaacs R. Differential Games // John Wiley and Sons, New York. (1965).
- [2]. Azamov A. A., Samatov B. T. The Π -strategy: analogies and applications. In: Proc. The Fourth Int. Conf. Game Theory and Management (GTM 2010), June 28-30, 2010, St. Petersburg, Russia, vol. 4. 2010. P. 33-47.
- [3]. Ibragimov G. I. A game of optimal pursuit of one object by several. J. Appl. Math. Mech., 1998. Vol. 62, No. 2, P. 187-192.
- [4]. Samatov B. T. The pursuit-evasion problem under integral-geometric constraints on pursuer controls. Autom. Remote Control, 2013. Vol. 74, No. 7. P. 1072-1081.
- [5]. Samatov, B.T., Soyibboev, U.B.: Differential game with a lifeline for the inertial movements of players. Ural Math. J. 7(2), 94-109 (2021).
- [6]. Samatov B.T. Horilov M.A., Jurayev B.I. Π -strategy for a differential game of pursuit with integral constraints of a generalized type // The Bulletin of Udmurt University. Mathematics. Mechanics. Comp. Science. (2023), 33(2), p. 293-311.
- [7]. Samatov, B., Soyibboev, U. Applications of the Π -Strategy When Players Move with Acceleration. In: Azimov, D. (eds) Proceedings of the IUTAM Symposium on Optimal Guidance and Control for Autonomous Systems 2023. IUTAM Bookseries, vol 40. Springer, Cham.
- [8]. Horilov M.A. On the “Life-line” problem when players move with the diminishing accelerations // Uzbek Mathematical Journal (2023), 67(2), p. 49-59.
- [9]. Bannikov A.S., Petrov N.N. Linear non-stationary differential pursuit games with several Evaders // Vestn. Udmurt. Univ., Mat. Mekh. Komp. Nauki. (2014), 25, p. 3-12.
- [10]. Alekseev, V.M., Tikhomirov, V.M., Fomin, S.V.: Optimal'noye upravleniye [Optimal Control]. Nauka, Moscow (1979).
- [11]. Samatov B.T. Horilov M.A., Akbarov A.Kh. Differential game: “Life line” for Non-Stationary Geometric Constraints on Controls // Lobachevskii Journal of Mathematics. (2022), 43(1), p. 237-248.

Boundary value problem with gluing conditions for an odd order equation with a discontinuous coefficient.

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Abstract: A Dirichlet-type problem is studied for a high-order odd equation with a discontinuous coefficient. Existence and uniqueness theorems are proved. A uniqueness criterion is obtained. The problem of "small denominators" arises during the study, and the "rate" of their approach to zero for certain values of the given problem parameters is determined.

Keywords: Equation, odd order, conjugation problem, "small denominator", existence, uniqueness.

I. INTRODUCTION

Boundary value problems for partial differential equations with discontinuous coefficients constitute an important area of modern mathematical analysis. Such equations arise naturally in the mathematical modelling of physical processes occurring in inhomogeneous media, where the physical properties of the medium, including density, viscosity, or other characteristics, may change abruptly across an interface. In particular, differential equations of odd order play a significant role in the study of wave propagation, hydrodynamic processes, filtration theory, and other applied problems [1–5].

A well-known example of an odd-order equation is the Korteweg–de Vries equation, which plays an important role in the theory of nonlinear waves and hydrodynamic phenomena [1,2]. Another important example is the viscous transonic equation describing axisymmetric transonic flows of a viscous gas. The theoretical investigation of such equations and their linearized forms has demonstrated the importance of studying boundary value problems for equations whose coefficients may possess discontinuities [3–5].

Boundary value problems for odd-order differential equations with smooth and continuous coefficients have been extensively investigated in the literature [6–11]. In addition, considerable attention has been devoted to degenerate higher-order equations [12] and to equations with discontinuous coefficients [13–17]. However, boundary value problems for high-order odd equations containing discontinuous coefficients require special analysis because the discontinuity changes the structure of the corresponding ordinary differential equations obtained after applying the method of separation of variables.

In the present study, a Dirichlet-type boundary value problem is considered for a high-order odd equation with a discontinuous coefficient in a rectangular domain. The discontinuity of the coefficient leads to different characteristic equations in the corresponding subdomains. Therefore, the construction of a solution requires the establishment of appropriate conjugation conditions at the line of discontinuity and the derivation of a unified system of algebraic equations for the unknown coefficients.

An important aspect of the problem is the investigation of the uniqueness and existence of its regular solution. By representing the solution as a Fourier series in the eigenfunctions of an appropriate one-dimensional problem, the original boundary value problem is reduced to a family of ordinary differential equations. The solvability of the resulting algebraic systems depends on the non-vanishing of their fundamental determinants. This, in turn, leads to a uniqueness criterion for the considered boundary value problem.

II. LITERATURE REVIEW

Boundary value problems for odd-order differential equations have been extensively studied because of their important theoretical and practical applications. The theory of nonlinear waves and integrable systems related to higher-order equations was developed in [1,2]. Problems arising in

transonic and hydrodynamic flows were investigated in [3–5]. Boundary value problems for third-order equations with multiple characteristics were studied in [6,7]. Degenerate high-order and high-odd-order equations were investigated in [8,12]. Other important results concerning the solvability of odd-order boundary value problems were obtained in [9–11]. Equations with discontinuous coefficients have also received considerable attention in [13–17]. In particular, the discontinuity of coefficients significantly affects the structure of characteristic equations and the corresponding solution methods. However, Dirichlet-type problems for high-order odd equations with discontinuous coefficients remain insufficiently investigated. Therefore, the present study focuses on establishing uniqueness and solvability conditions and overcoming the small-denominator problem.

1. Problem Statement and Introduction. In the domain $\Omega = \{(x, y) : -a < x < a, 0 < y < 1\}$, $a \in \mathbb{R}$, consider the following problem.

Problem D. Find the function

$$u(x, y) \in C_{x,y}^{4n-1,2m}(\Omega \setminus \{x=0\}) \cap C_{x,y}^{4n-2,2m-1}(\bar{\Omega}),$$

where $C_{x,y}^{p,q}(\Omega)$ denotes the class of functions continuous in Ω together with all partial derivatives

of the form $\partial^{r+s}u / \partial x^r \partial y^s$ for $r \leq p$, $s \leq q$; such that the derivative $\frac{\partial^{4n-1}u}{\partial x^{4n-1}}$ has finite one-sided limits as $x \rightarrow \pm 0$ for each $y \in (0,1)$, and the following conditions are satisfied

$$\left\{ \begin{array}{l} \frac{\partial^{4n-1}u}{\partial x^{4n-1}} + \operatorname{sgn}(x)(-1)^m \frac{\partial^{2m}u}{\partial y^{2m}} = 0, \quad x \neq 0; \\ \frac{\partial^j u(a, y)}{\partial x^j} = \varphi_j(y), \quad \frac{\partial^j u(-a, y)}{\partial x^j} = \psi_j(y), \quad j = 0, 1, \dots, 2n-2; \\ \frac{\partial^{2n-1}u(a, y)}{\partial x^{2n-1}} = \frac{\partial^{2n-1}u(-a, y)}{\partial x^{2n-1}}; \\ \frac{\partial^s u(+0, y)}{\partial x^s} = \frac{\partial^s u(-0, y)}{\partial x^s}, \quad s = 0, 1, \dots, 4n-2; \\ \frac{\partial^{2t}u(x, 0)}{\partial y^{2t}} = \frac{\partial^{2t}u(x, 1)}{\partial y^{2t}} = 0, \quad t = 0, 1, \dots, m-1, \end{array} \right. \quad (1)$$

where $n, m \in \mathbb{N}$, $\gcd(4n-1, 2m) = 1$; $\varphi_j(y), \psi_j(y)$ are given continuous functions.

A note on the role of the condition $\gcd(4n-1, 2m) = 1$. This condition guarantees that the number $\frac{2m}{4n-1}$ is an irreducible fraction. This is essential when calculating the $(4n-1)$ -th

roots of $(\pi k)^{2m}$: when $\gcd(4n-1, 2m) = 1$, the characteristic exponent $\mu_k = (\pi k)^{2m/(4n-1)}$ cannot be represented as a rational factor of πk , which ensures the pairwise distinctness of the roots of the characteristic equation and, consequently, the non-degeneracy of the system.

Odd-order equations with discontinuous coefficients are of considerable interest from both theoretical and applied perspectives. They arise in the mathematical modeling of wave processes in inhomogeneous media, in problems of filtration theory, and in hydrodynamics when describing flows with discontinuous physical characteristics (density, viscosity). Odd-order equations have

numerous applications in applied problems. For example, the Korteweg–de Vries equation is a third-order nonlinear partial differential equation that plays an important role in the theory of nonlinear waves, primarily of hydrodynamic origin [1,2]. The equation has the form:

$$u_t(x, t) + 6uu_x + u_{xxx} = 0.$$

Another classic example is the viscous transonic equation (VT-equation), which describes axisymmetric transonic flows of a viscous gas. The general theory of short waves and its relationship to transonic flows is developed in [3].

The asymptotic picture of the sonic flow of a viscous heat-conducting gas around bodies of revolution was studied in [4]. The VT equation for the perturbed potential function $\varphi(x, y)$ has the form [5]:

$$\varphi_{xxx}(x, y) + \varphi_{yy} - \frac{\nu}{y}\varphi_y = \varphi_x \varphi_{xx}.$$

The linearized VT equation coincides (for $\nu = 0$) with $\varphi_{xxx} + \varphi_{yy} = 0$, a special case of the equation of Problem D for $x > 0$. The Green's function of the linearized VT equation was studied in [5].

Regarding theoretical studies of boundary value problems for odd-order equations, it should be noted that equations with smooth and continuous coefficients have been studied primarily, for example [6]–[11]. Degenerate equations have also been investigated, for example [12]. Boundary value problems for third-order equations with discontinuous coefficients of the first derivative were studied in [13]–[15] and elsewhere. Even-order equations with discontinuous coefficients were studied, for example, in [16],[17]. The present paper is devoted to the study of a boundary value problem with a discontinuous coefficient of the derivative with respect to y .

A criterion for the uniqueness of the solution is proved in this paper. The solution is constructed as a series in eigenfunctions of a one-dimensional problem. The study raises the classical problem of "small denominators," which is successfully overcome.

2. Main Part. We will formally seek a solution to Problem D in the form

$$u(x, y) = \sum_{k=1}^{\infty} X_k(x) \sin \pi k y. \quad (2)$$

The system of functions $\{\sin \pi k y\}_{k=1}^{\infty}$ is orthogonal and complete in $L_2(0, 1)$; the conditions on y from (1) are satisfied for each term. Formal substitution of series (2) into equation (1) and use of orthogonality lead to the following boundary value problem for $X_k(x)$:

$$\begin{cases} X_k^{(4n-1)}(x) + (\pi k)^{2m} X_k(x) = 0, & 0 < x < a; \\ X_k^{(4n-1)}(x) - (\pi k)^{2m} X_k(x) = 0, & -a < x < 0; \\ X_k^{(j)}(a) = \varphi_{jk}, \quad X_k^{(j)}(-a) = \psi_{jk}, & j = 0, 1, \dots, 2n-2; \\ X_k^{(2n-1)}(a) = X_k^{(2n-1)}(-a); \\ X_k^{(s)}(+0) = X_k^{(s)}(-0), & s = 0, 1, \dots, 4n-2, \end{cases} \quad (3)$$

where the Fourier coefficients of the boundary data are equal to

$$\varphi_{jk} = 2 \int_0^1 \varphi_j(y) \sin \pi k y dy, \quad \psi_{jk} = 2 \int_0^1 \psi_j(y) \sin \pi k y dy, \quad j = 0, 1, \dots, 2n-2; k = 1, 2, \dots$$

The validity of the substitution and the uniform convergence of series (2) after finding X_k are established in Theorems 3 and 4.

General solution for $x > 0$. The characteristic equation $\lambda^{4n-1} + (\pi k)^{2m} = 0$ has exactly $4n-1$ roots. We denote $\mu_k = (\pi k)^{2m/(4n-1)} > 0$. Then the roots are $\lambda = \mu_k e^{i\theta}$, where $e^{i(4n-1)\theta} = -1 = e^{i\pi}$, whence

$$\theta_s = \frac{2s+1}{4n-1}\pi, \quad s = 0, 1, \dots, 4n-2.$$

For $s = 2n-1$, we have $\theta_{2n-1} = \pi$, that is, $\lambda_{2n-1} = -\mu_k$ — the only real root (it is negative). We denote $\alpha_s = \cos \theta_s$, $\beta_s = \sin \theta_s$ for $s = 0, \dots, 2n-2$. For s in this range, $\alpha_s = \cos \frac{(2s+1)\pi}{4n-1}$; for $s = 0, \dots, n-1$, we have $\theta_s \in (0, \pi/2)$, so $\alpha_s > 0$; for $s = n, \dots, 2n-2$ we have $\theta_s \in (\pi/2, \pi)$, so $\alpha_s < 0$.

General solution for $x > 0$:

$$X_k(x) = \sum_{s=0}^{2n-2} e^{\mu_k \alpha_s x} (c_s^1 \cos \mu_k \beta_s x + c_s^2 \sin \mu_k \beta_s x) + c_{2n-1} e^{-\mu_k x}. \quad (4)$$

Let's show that

$$X_k^{(j)}(x) = \mu_k^j \left(\sum_{s=0}^{2n-2} e^{\mu_k \alpha_s x} (c_s^1 \cos(\mu_k \beta_s x + j\theta_s) + c_s^2 \sin(\mu_k \beta_s x + j\theta_s)) + (-1)^j c_{2n-1} e^{-\mu_k x} \right). \quad (5)$$

We have

$$\begin{aligned} X_k'(x) &= \sum_{s=0}^{2n-2} \mu_k \alpha_s e^{\mu_k \alpha_s x} (c_s^1 \cos \mu_k \beta_s x + c_s^2 \sin \mu_k \beta_s x) + (-\mu_k) c_{2n-1} e^{-\mu_k x} + \\ &\quad + \sum_{s=0}^{2n-2} e^{\mu_k \alpha_s x} (-\mu_k \beta_s c_s^1 \sin \mu_k \beta_s x + \mu_k \beta_s c_s^2 \cos \mu_k \beta_s x) = \\ &= \sum_{s=0}^{2n-2} \mu_k e^{\mu_k \alpha_s x} (c_s^1 \alpha_s \cos \mu_k \beta_s x - \beta_s c_s^1 \sin \mu_k \beta_s x + \alpha_s c_s^2 \sin \mu_k \beta_s x + \beta_s c_s^2 \cos \mu_k \beta_s x) + \\ &\quad + (-\mu_k) c_{2n-1} e^{-\mu_k x} = \\ &= \sum_{s=0}^{2n-2} \mu_k e^{\mu_k \alpha_s x} (c_s^1 \cos(\mu_k \beta_s x + \theta_s) + c_s^2 \sin(\mu_k \beta_s x + \theta_s)) + (-\mu_k) c_{2n-1} e^{-\mu_k x}, \end{aligned}$$

Differentiating j once representation (4), we obtain formula (5).

Let's introduce the following notation:

$$T_{js}^1(x) = e^{\mu_k \alpha_s x} \cos(\mu_k \beta_s x + j\theta_s), \quad T_{js}^2(x) = e^{\mu_k \alpha_s x} \sin(\mu_k \beta_s x + j\theta_s).$$

Then formula (5) can be written in the form

$$X_k^{(j)}(x) = \mu_k^j \left(\sum_{s=0}^{2n-2} (c_s^1 T_{js}^1(x) + c_s^2 T_{js}^2(x)) + (-1)^j c_{2n-1} e^{-\mu_k x} \right), \quad j = 0, 1, \dots, 4n-1. \quad (6)$$

General solution for $x < 0$. For $x < 0$, the equation has the form $X_k^{(4n-1)} - (\pi k)^{2m} X_k = 0$, that is, the characteristic equation is $\lambda^{4n-1} = (\pi k)^{2m} = \mu_k^{4n-1}$, the roots are: $\lambda = \mu_k e^{2\pi i s/(4n-1)}$, $s = 0, \dots, 4n-2$. For $s = 0$, we obtain $\lambda_0 = \mu_k > 0$, the only real root.

Let us denote $\tilde{\alpha}_s = \cos \frac{2\pi s}{4n-1}$, $\tilde{\beta}_s = \sin \frac{2\pi s}{4n-1}$ for $s = 1, \dots, 2n-1$. General solution:

$$X_k(x) = d_0 e^{\mu_k x} + \sum_{s=1}^{2n-1} e^{\mu_k \tilde{\alpha}_s x} (d_s^1 \cos \mu_k \tilde{\beta}_s x + d_s^2 \sin \mu_k \tilde{\beta}_s x).$$

Using the substitution $x \mapsto -x$ (that is, denoting $\xi = -x > 0$), and relating the roots of λ to the roots of the problem for $x > 0$ via the relation $\lambda^{4n-1} = \mu_k^{4n-1} = (-\mu_k)^{4n-1} \cdot (-1)^{4n-1}$, it can be shown that the derivatives for $x < 0$ are expressed by the formula:

$$X_k^{(j)}(x) = (-1)^j \mu_k^j \left(\sum_{s=0}^{2n-2} (d_s^1 T_{js}^1(-x) + d_s^2 T_{js}^2(-x)) + (-1)^j d_{2n-1} e^{\mu_k x} \right), \quad (7)$$

where $d_s^1, d_s^2, d_{2n-1} = \text{const}$.

Having satisfied the conditions of problem (1) and using (6), (7) we obtain a system of algebraic equations for the unknowns $c_s^1, c_s^2, c_{2n-1}, d_s^1, d_s^2, d_{2n-1}$. Let's write the system explicitly:

$$\left\{ \begin{array}{l} \sum_{s=0}^{2n-2} (c_s^1 T_{js}^1(0) + c_s^2 T_{js}^2(0)) - c_{2n-1} + \sum_{s=0}^{2n-2} (d_s^1 T_{js}^1(0) + d_s^2 T_{js}^2(0)) - d_{2n-1} = 0, \\ j = 1, 3, \dots, 4n-3, \\ \sum_{s=0}^{2n-2} (c_s^1 T_{js}^1(0) + c_s^2 T_{js}^2(0)) + c_{2n-1} - \sum_{s=0}^{2n-2} (d_s^1 T_{js}^1(0) + d_s^2 T_{js}^2(0)) - d_{2n-1} = 0, \\ j = 0, 2, \dots, 4n-2, \\ \sum_{s=0}^{2n-2} (c_s^1 T_{js}^1(a) + c_s^2 T_{js}^2(a)) + (-1)^j c_{2n-1} e^{-\mu_k a} = \frac{\varphi_{jk}}{\mu_k^j}, \quad j = 0, 1, \dots, 2n-2, \\ \sum_{s=0}^{2n-2} (d_s^1 T_{js}^1(a) + d_s^2 T_{js}^2(a)) + (-1)^j d_{2n-1} e^{-\mu_k a} = \frac{(-1)^j \psi_{jk}}{\mu_k^j}, \quad j = 0, 1, \dots, 2n-2, \\ \sum_{s=0}^{2n-2} (c_{2n-1,s}^1 T_{js}^1(a) + c_{2n-1,s}^2 T_{js}^2(a)) + c_{2n-1} e^{-\mu_k a} \\ + \sum_{s=0}^{2n-2} (d_{2n-1,s}^1 T_{js}^1(a) + d_{2n-1,s}^2 T_{js}^2(a)) + d_{2n-1} e^{-\mu_k a} = 0. \end{array} \right. \quad (8)$$

Let us calculate the main determinant $\Delta = \Delta(a, k, n, m)$ of system (8). In block notation:

$$\Delta = \begin{vmatrix} A_{2n-1,4n-2} & -1 & A_{2n-1,4n-2} & -1 \\ B_{2n,4n-2} & 1 & -B_{2n,4n-2} & -1 \\ C_{2n-1,4n-2} & (-1)^j e^{-\mu_k a} & 0 & 0 \\ 0 & 0 & C_{2n-1,4n-2} & (-1)^j e^{-\mu_k a} \\ D_{1,4n-2} & e^{-\mu_k a} & D_{1,4n-2} & e^{-\mu_k a} \end{vmatrix},$$

where blocks A, B, C, D consist of the values of $T_{js}^{1,2}$ at $x=0$ and $x=a$, respectively. After some elementary transformations and applying the block decomposition, we obtain

$$\Delta = M \begin{vmatrix} C_{2n-1,4n-2} & (-1)^j e^{-\mu_k a} \\ B_{2n,4n-2} & 1 \end{vmatrix} \begin{vmatrix} A_{2n-1,4n-2} & -1 \\ C_{2n-1,4n-2} & (-1)^j e^{-\mu_k a} \\ D_{1,4n-2} & e^{-\mu_k a} \end{vmatrix}, M = \text{const}, \quad (9)$$

where

$$\begin{aligned}
|A_{2n-1,4n-2} \quad -1| &= \begin{vmatrix} T_{10}^1(0) & T_{10}^2(0) & \cdots & T_{1,2n-2}^1(0) & T_{1,2n-2}^2(0) & -1 \\ T_{30}^1(0) & T_{30}^2(0) & \cdots & T_{3,2n-2}^1(0) & T_{3,2n-2}^2(0) & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ T_{4n-3,0}^1(0) & \cdots & \cdots & T_{4n-3,2n-2}^1(0) & T_{4n-3,2n-2}^2(0) & -1 \end{vmatrix}, \\
|B_{2n,4n-2} \quad 1| &= \begin{vmatrix} T_{00}^1(0) & T_{00}^2(0) & \cdots & T_{0,2n-2}^1(0) & T_{0,2n-2}^2(0) & 1 \\ T_{20}^1(0) & T_{20}^2(0) & \cdots & T_{2,2n-2}^1(0) & T_{2,2n-2}^2(0) & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ T_{4n-2,0}^1(0) & T_{4n-2,0}^2(0) & \cdots & T_{4n-2,2n-2}^1(0) & T_{4n-2,2n-2}^2(0) & 1 \end{vmatrix}, \\
|C_{2n-1,4n-2} \quad (-1)^j e^{-\mu_k a}| &= \begin{vmatrix} T_{00}^1(a) & T_{00}^2(a) & \cdots & T_{0,2n-2}^1(a) & T_{0,2n-2}^2(a) & e^{-\mu_k a} \\ T_{10}^1(a) & T_{10}^2(a) & \cdots & T_{1,2n-2}^1(a) & T_{1,2n-2}^2(a) & -e^{-\mu_k a} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ T_{2n-2,0}^1(a) & \cdots & \cdots & T_{2n-2,2n-2}^1(a) & T_{2n-2,2n-2}^2(a) & (-1)^{2n-2} e^{-\mu_k a} \end{vmatrix}, \\
|D_{1,4n-2} \quad e^{-\mu_k a}| &= \begin{vmatrix} T_{2n-1,0}^1(a) & T_{2n-1,0}^2(a) & \cdots & T_{2n-1,2n-2}^1(a) & T_{2n-1,2n-2}^2(a) & e^{-\mu_k a} \end{vmatrix}.
\end{aligned}$$

Given the completeness of the system $\{\sin \pi k y\}_{k=1}^{\infty}$ in $L_2(0,1)$, we obtain the following criterion.

Theorem 1. {(Uniqueness criterion).} A solution to problem D is unique if and only if the fundamental determinant Δ of system (8) is nonzero for all $k \in \mathbb{N}$.

Now, assuming $\Delta \neq 0$, we consider each factor in (9) and find the asymptotic of Δ as $k \rightarrow +\infty$. Consider the first factor:

$$\begin{aligned}
\Delta_2 &= \begin{vmatrix} C_{2n-1,4n-2} & (-1)^j e^{-\mu_k a} \\ B_{2n,4n-2} & 1 \end{vmatrix}_{(4n-1) \times (4n-1)} = \\
&= \begin{vmatrix} T_{00}^1(a) & T_{00}^2(a) & \cdots & T_{0,2n-2}^1(a) & T_{0,2n-2}^2(a) & e^{-\mu_k a} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ T_{2n-2,0}^1(a) & \cdots & \cdots & T_{2n-2,2n-2}^1(a) & T_{2n-2,2n-2}^2(a) & (-1)^{2n-2} e^{-\mu_k a} \\ T_{00}^1(0) & T_{00}^2(0) & \cdots & T_{0,2n-2}^1(0) & T_{0,2n-2}^2(0) & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ T_{4n-2,0}^1(0) & T_{4n-2,0}^2(0) & \cdots & T_{4n-2,2n-2}^1(0) & T_{4n-2,2n-2}^2(0) & 1 \end{vmatrix} =
\end{aligned}$$

$$= \left(\frac{i}{2} \right)^{2n-1} \times \begin{vmatrix} e^{\mu_k \alpha_0 a + i(\mu_k \beta_0 a + 0^* \theta_0)} & e^{\mu_k \alpha_0 a - i(\mu_k \beta_0 a + 0^* \theta_0)} & \dots & . & e^{\mu_k \alpha_{2n-2} a - i(\mu_k \beta_{2n-2} a + 0^* \theta_{2n-2})} & e^{-\mu_k a} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ e^{\mu_k \alpha_0 a + i(\mu_k \beta_0 a + (2n-2)\theta_0)} & e^{\mu_k \alpha_0 a - i(\mu_k \beta_0 a + (2n-2)\theta_0)} & \dots & . & e^{\mu_k \alpha_{2n-2} a - i(\mu_k \beta_{2n-2} a + (2n-2)\theta_{2n-2})} & e^{-\mu_k a} \\ e^{i(0^* \theta_0)} & e^{-i(0^* \theta_0)} & \dots & . & e^{-i(0^* \theta_{2n-2})} & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ e^{i(4n-2)\theta_0} & e^{-i(4n-2)\theta_0} & \dots & . & e^{-i(4n-2)\theta_{2n-2}} & 1 \end{vmatrix},$$

For $k \rightarrow +\infty$, the terms with $e^{\mu_k \alpha_s a}$ for $\alpha_s > 0$ (that is, $s = 0, \dots, n-1$) dominate. The leading term of the asymptotic (9) is determined by the expansion in these "exponentially growing" columns and the "purely trigonometric" block for $x = 0$, denoted by Δ_3 . We have

$$\Delta_3 = \left(\frac{i}{2} \right)^{2n-1} (A_1 B_1 - A_2 B_2),$$

$$A_{1(2n-1 \times 2n-1)} = e^{\mu_k a \left(2 \sum_{j=0}^{n-2} \alpha_j + \alpha_{n-1} \right)} e^{i \mu_k \beta_{n-1} a} \times \begin{vmatrix} 1 & 1 & . & 1 & 1 & 1 \\ e^{i\theta_0} & e^{-i\theta_0} & . & e^{i\theta_{n-2}} & e^{-i\theta_{n-2}} & e^{i\theta_{n-1}} \\ . & . & . & . & . & . \\ e^{i(2n-2)\theta_0} & e^{-i(2n-2)\theta_0} & . & e^{i(2n-2)\theta_{n-2}} & e^{-i(2n-2)\theta_{n-2}} & e^{i(2n-2)\theta_{n-1}} \end{vmatrix},$$

$$B_{1(2n \times 2n)} = \begin{vmatrix} 1 & 1 & . & 1 & 1 & 1 \\ e^{-2i\theta_{n-1}} & e^{2i\theta_n} & . & e^{2i\theta_{2n-2}} & e^{-2i\theta_{2n-2}} & 1 \\ . & . & . & . & . & . \\ e^{-i(4n-2)\theta_{n-1}} & e^{i(4n-2)\theta_n} & . & e^{i(4n-2)\theta_{n-2}} & e^{-i(4n-2)\theta_{2n-2}} & 1 \end{vmatrix},$$

$$A_2 = e^{\mu_k a \left(2 \sum_{j=0}^{n-2} \alpha_j + \alpha_{n-1} \right)} e^{-i \mu_k \beta_{n-1} a} \times$$

$$\begin{vmatrix} 1 & 1 & \cdot & 1 & 1 & 1 \\ e^{i\theta_0} & e^{-i\theta_0} & \cdot & e^{i\theta_{n-2}} & e^{-i\theta_{n-2}} & e^{-i\theta_{n-1}} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ e^{i(2n-2)\theta_0} & e^{-i(2n-2)\theta_0} & \cdot & e^{i(2n-2)\theta_{n-2}} & e^{-i(2n-2)\theta_{n-2}} & e^{-i(2n-2)\theta_{n-1}} \end{vmatrix},$$

$$B_2 = \begin{vmatrix} 1 & 1 & \cdot & 1 & 1 & 1 \\ e^{2i\theta_{n-1}} & e^{2i\theta_n} & \cdot & e^{2i\theta_{2n-2}} & e^{-2i\theta_{2n-2}} & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ e^{i(4n-2)\theta_{n-1}} & e^{i(4n-2)\theta_n} & \cdot & e^{i(4n-2)\theta_{n-2}} & e^{-i(4n-2)\theta_{2n-2}} & 1 \end{vmatrix}.$$

Now we calculate Δ_3 . To do this, we denote the common factor by M and factor it out of the brackets.

$$A_1 = M e^{\mu_k a \left(2 \sum_{j=0}^{n-2} \alpha_j + \alpha_{n-1} \right)} e^{i \mu_k \beta_{n-1} a} \prod_{j=0}^{n-2} (e^{i\theta_{n-1}} - e^{i\theta_j}) (e^{i\theta_{n-1}} - e^{-i\theta_j}),$$

$$B_1 = M \prod_{j=n}^{2n-2} (e^{-2i\theta_{n-1}} - e^{2i\theta_j}) (e^{-2i\theta_{n-1}} - e^{-2i\theta_j}) (e^{-2i\theta_{n-1}} - 1),$$

$$A_2 = M e^{\mu_k a \left(2 \sum_{j=0}^{n-2} \alpha_j + \alpha_{n-1} \right)} e^{-i \mu_k \beta_{n-1} a} \prod_{j=0}^{n-2} (e^{-i\theta_{n-1}} - e^{i\theta_j}) (e^{-i\theta_{n-1}} - e^{-i\theta_j}),$$

$$B_2 = M \prod_{j=n}^{2n-2} (e^{2i\theta_{n-1}} - e^{2i\theta_j}) (e^{2i\theta_{n-1}} - e^{-2i\theta_j}) (e^{2i\theta_{n-1}} - 1),$$

further

$$\prod_{j=0}^{n-2} (e^{i\theta_{n-1}} - e^{i\theta_j}) (e^{i\theta_{n-1}} - e^{-i\theta_j}) = \prod_{j=0}^{n-2} (e^{2i\theta_{n-1}} - e^{i\theta_j + i\theta_{n-1}} - e^{i\theta_{n-1} - i\theta_j} + 1) =$$

$$= \prod_{j=0}^{n-2} (e^{i\theta_{n-1}} (e^{i\theta_{n-1}} + e^{-i\theta_{n-1}}) - e^{i\theta_{n-1}} (e^{i\theta_j} + e^{-i\theta_j})) = \prod_{j=0}^{n-2} (2e^{i\theta_{n-1}} \cos \theta_{n-1} - 2e^{i\theta_{n-1}} \cos \theta_j) =$$

$$= 2^{n-1} e^{i(n-1)\theta_{n-1}} \prod_{j=0}^{n-2} (\cos \theta_{n-1} - \cos \theta_j) = M e^{i(n-1)\theta_{n-1}},$$

similarly

$$\prod_{j=0}^{n-2} (e^{-i\theta_{n-1}} - e^{i\theta_j}) (e^{-i\theta_{n-1}} - e^{-i\theta_j}) = 2^{n-1} e^{-i(n-1)\theta_{n-1}} \prod_{j=0}^{n-2} (\cos \theta_{n-1} - \cos \theta_j) = M e^{-i(n-1)\theta_{n-1}},$$

further

$$\prod_{j=n}^{2n-2} (e^{-2i\theta_{n-1}} - e^{2i\theta_j}) (e^{-2i\theta_{n-1}} - e^{-2i\theta_j}) (e^{-2i\theta_{n-1}} - 1) =$$

$$\begin{aligned}
&= \prod_{j=n}^{2n-2} \left(e^{-4i\theta_{n-1}} - e^{2i\theta_j - 2i\theta_{n-1}} - e^{-2i\theta_{n-1} - 2i\theta_j} + 1 \right) (e^{-2i\theta_{n-1}} - 1) = \\
&= \prod_{j=n}^{2n-2} \left(e^{-2i\theta_{n-1}} (e^{-2i\theta_{n-1}} + e^{2i\theta_{n-1}}) - e^{-2i\theta_{n-1}} (e^{2i\theta_j} + e^{-2i\theta_j}) \right) (e^{-2i\theta_{n-1}} - 1) = \\
&= 2^{n-1} e^{-2i(n-1)\theta_{n-1}} \prod_{j=n}^{2n-2} (\cos 2\theta_{n-1} - \cos 2\theta_j) (e^{-2i\theta_{n-1}} - 1) = \\
&= 2^{n-1} e^{-2i(n-1)\theta_{n-1}} e^{-i\theta_{n-1}} (e^{-i\theta_{n-1}} - e^{i\theta_{n-1}}) \prod_{j=n}^{2n-2} (\cos 2\theta_{n-1} - \cos 2\theta_j) = \\
&= 2^{n-1} e^{-2i(n-1)\theta_{n-1}} e^{-i\theta_{n-1}} (-2i \sin \theta_{n-1}) \prod_{j=n}^{2n-2} (\cos 2\theta_{n-1} - \cos 2\theta_j) = \\
&= Me^{-2i(n-1)\theta_{n-1}} e^{-i\theta_{n-1}} (-i \sin \theta_{n-1}),
\end{aligned}$$

similarly

$$\begin{aligned}
&\prod_{j=n}^{2n-2} (e^{2i\theta_{n-1}} - e^{2i\theta_j}) (e^{2i\theta_{n-1}} - e^{-2i\theta_j}) (e^{2i\theta_{n-1}} - 1) = \\
&= Me^{2i(n-1)\theta_{n-1}} e^{i\theta_{n-1}} (i \sin \theta_{n-1}).
\end{aligned}$$

Substituting, we get

$$\begin{aligned}
\Delta_3 &= Me^{\mu_k a \left(2 \sum_{j=0}^{n-2} \alpha_j + \alpha_{n-1} \right)} \left(e^{i\mu_k \beta_{n-1} a} e^{i(n-1)\theta_{n-1}} e^{-2i(n-1)\theta_{n-1}} e^{-i\theta_{n-1}} + \right. \\
&\quad \left. + e^{-i\mu_k \beta_{n-1} a} e^{-i(n-1)\theta_{n-1}} e^{2i(n-1)\theta_{n-1}} e^{i\theta_{n-1}} \right) = \\
&= Me^{\mu_k a \left(2 \sum_{j=0}^{n-2} \alpha_j + \alpha_{n-1} \right)} \left(e^{i(\mu_k \beta_{n-1} a - n\theta_{n-1})} + e^{-i(\mu_k \beta_{n-1} a - n\theta_{n-1})} \right) = \\
&= Me^{\mu_k a \left(2 \sum_{j=0}^{n-2} \alpha_j + \alpha_{n-1} \right)} \cos(\mu_k \beta_{n-1} a - n\theta_{n-1}) = \\
&= Me^{\mu_k a \left(2 \sum_{j=0}^{n-2} \alpha_j + \alpha_{n-1} \right)} \cos \left(a\mu_k \sin \left(\frac{2n-1}{4n-1} \pi \right) - \frac{n(2n-1)}{4n-1} \pi \right) = \\
&= Me^{\mu_k a \left(2 \sum_{j=0}^{n-2} \alpha_j + \alpha_{n-1} \right)} \sin \pi \left(Ck^{2m/(4n-1)} - \frac{4n^2 + 1 - 6n}{2(4n-1)} \right), \\
&\quad C = a\pi^{2m/(4n-1)-1} \sin \left(\frac{2n-1}{4n-1} \pi \right).
\end{aligned}$$

Let's consider the second factor in expression (9), denote it by Δ_4 . We denote the leading term of the asymptotic by Δ_5 . We have

$$\Delta_4 = O(\Delta_5), \Delta_5 = \left| \frac{C_{2n-1, 2n}}{D_{1, 2n}} \right| |A_{2n-1, 2n-2} - 1|,$$

where

$$\begin{aligned}
& \left| \begin{matrix} C_{2n-1,2n} \\ D_{1,2n} \end{matrix} \right| = \begin{vmatrix} T_{00}^1(a) & T_{00}^2(a) & \dots & T_{0,n-1}^1(a) & T_{0,n-1}^2(a) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ T_{2n-2,0}^1(a) & T_{2n-2,0}^2(a) & \dots & T_{2n-2,n-1}^1(a) & T_{2n-2,n-1}^2(a) \\ T_{2n-1,0}^1(a) & T_{2n-1,0}^2(a) & \dots & T_{2n-1,n-1}^1(a) & T_{2n-1,n-1}^2(a) \end{vmatrix} = \\
& = Me^{2\mu_k a \sum_{j=0}^{n-1} \alpha_j} \begin{vmatrix} e^{i\theta_0} & e^{-i\theta_0} & \dots & e^{i\theta_{n-1}} & e^{-i\theta_{n-1}} \\ e^{3i\theta_0} & e^{-3i\theta_0} & \cdot & e^{3i\theta_{n-1}} & e^{-3i\theta_{n-1}} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ e^{(4n-3)i\theta_0} & e^{-(4n-3)i\theta_0} & \dots & e^{(4n-3)i\theta_{n-1}} & e^{-(4n-3)i\theta_{n-1}} \end{vmatrix} = \\
& = Me^{2\mu_k a \sum_{j=0}^{n-1} \alpha_j}, M = \text{const} \neq 0, \\
& |A_{2n-1,2n-2} - 1| = \\
& = \begin{vmatrix} T_{1,n}^1(0) & T_{1,n}^2(0) & \dots & T_{1,2n-2}^1(0) & T_{1,2n-2}^2(0) & -1 \\ T_{3,n}^1(0) & T_{3,n}^2(0) & \dots & T_{3,2n-2}^1(0) & T_{3,2n-2}^2(0) & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ T_{4n-3,n}^1(0) & T_{4n-3,n}^2(0) & \dots & T_{4n-3,2n-2}^1(0) & T_{4n-3,2n-2}^2(0) & -1 \end{vmatrix} = M = \text{const} \neq 0, \\
& \Delta_4 = Me^{2\mu_k a \sum_{j=0}^{n-1} \alpha_j} + o\left(e^{2\mu_k a \sum_{j=0}^{n-1} \alpha_j}\right),
\end{aligned}$$

so

$$\Delta = Me^{\mu_k a \left(4 \sum_{j=0}^{n-2} \alpha_j + 3\alpha_{n-1}\right)} \sin \pi \left(Ck^{\frac{2m}{4n-1}} - \frac{4n^2 + 1 - 6n}{2(4n-1)} \right) + o\left(e^{\mu_k a \left(4 \sum_{j=0}^{n-2} \alpha_j + 3\alpha_{n-1}\right)}\right). \quad (10)$$

Now, assuming $\Delta \neq 0$ and solving system (8) using Cramer's method, and then substituting the resulting $c_s^1, c_s^2, d_s^1, d_s^2$ into expressions (6), (7), we obtain the following estimate.

Theorem 2. Under the conditions of Theorem 1, the following estimates hold: for all $j = 0, 1, \dots, 4n-1$ and $x \in [-a, a]$

$$\left| \frac{d^j X_k(x)}{dx^j} \right| \leq \frac{Mk^{\frac{2m}{4n-1}j} \sum_{s=0}^{2n-1} (|\varphi_{sk}| + |\psi_{sk}|)}{\left| \sin \left(\pi \left\{ Ck^{\frac{2m}{4n-1}} - \frac{4n^2 - 6n + 1}{2(4n-1)} \right\} \right) + o(1) \right|}, \quad k \rightarrow +\infty;$$

here $0 < M = \text{const}$, M is independent of k, j, x .

Lemma 1. There exist infinitely many values of $a > 0$ such that $\Delta(a, k) \neq 0$ for all $k \in \mathbb{N}$.

Proof. Fix k and consider the equation $\Delta(a, k) = 0$ as an equation with respect to a . The function $\Delta(a, k)$ for fixed k is an entire function of a (all the input expressions are sums of exponentials and trigonometric functions of $\mu_k a$, which are entire functions of a). Moreover, $\Delta(\cdot, k) \not\equiv 0$: indeed, the asymptotics show that $\Delta(a, k) \neq 0$ for sufficiently small a . Since $\Delta(\cdot, k)$ is an entire function that is not identical to zero, the set of its real zeros $S_k := \{a \in \mathbb{R} : \Delta(a, k) = 0\}$ is at most countable. Consider the union

$$S = \bigcup_{k=1}^{\infty} S_k.$$

A countable union of countable sets is countable. Therefore, S is a countable subset of $(0, +\infty)$, and there exists an uncountable (in particular, infinite) set of values $a \in (0, +\infty) \setminus S$ such that $\Delta(a, k) \neq 0$ for all $k \in \mathbb{N}$.

Lemma 1 is proved.

Note that even if the conditions of Theorem 1 are met, the problem of “small denominators” in expression (10) may arise. Let us address this issue.

Lemma 2. Let the following conditions be satisfied: a) Theorem 1 is true; b) $C \in \mathbb{R}$, $C^{4n-1} = p/q$, where

$$p, q \in P(n) = \left\{ p, q \in \mathbb{N}, (p, q) = 1 \mid \forall s, t \in \mathbb{N} : p \cdot (2(4n-1))^{4n-1} s^{2m} \neq q \cdot ((4n^2 - 6n + 1) + 2t(4n-1))^{4n-1} \right\},$$

then

$$\left| \sin \left(\pi \left\{ Ck^{\frac{2m}{4n-1}} - \frac{4n^2 - 6n + 1}{2(4n-1)} \right\} \right) \right| \geq \frac{M}{k^{\frac{4m(2n-1)}{4n-1}}}, \quad 0 < M = \text{const}, \quad \forall k \in \mathbb{N}.$$

Proof. First, we show that $P(n) \neq \emptyset$. Let's take $C = 2^{1/(4n-1)}$, then $C^{4n-1} = 2$, $p = 2$, $q = 1$. The condition $p, q \in P(n)$ takes the form: $2 \cdot (2(4n+1))^{4n+1} s^{2m} \neq ((4n^2 - 2n - 1) + 2t(4n+1))^{4n+1}$ for all $s, t \in \mathbb{N}$. The last inequality is true because the left side is even, while the right is odd. Let us use the inequality $|\sin \pi x| \geq 2 \|x\|$, where $\|x\| = \min_{t \in \mathbb{Z}} |x - t|$.

Let $z_k = Ck^{\frac{2m}{4n-1}} - \frac{4n^2 - 6n + 1}{2(4n-1)}$ and choose $t_k \in \mathbb{Z}$ so that $\|z_k\| = |z_k - t_k|$. Then

$$|z_k - t_k| = \frac{|2(4n-1)Ck^{\frac{2m}{4n-1}} - ((4n^2 - 6n + 1) + 2t_k(4n-1))|}{2(4n-1)}.$$

Since $C^{4n-1} = p/q$, we have $(2(4n-1)C)^{4n-1} = p \cdot \frac{(2(4n-1))^{4n-1}}{q}$. We multiply the numerator and denominator by q and use the identity

$$a - b = \frac{a^{4n-1} - b^{4n-1}}{a^{4n-2} + a^{4n-3}b + \dots + b^{4n-2}}$$

with $a = 2(4n-1)Ck^{\frac{2m}{4n-1}}$ and $b = (4n^2 - 6n + 1 + 2t_k(4n-1))$:

$$|z_k - t_k| = \frac{|p(2(4n-1))^{4n-1} k^{2m} - q(4n^2 - 6n + 1 + 2t_k(4n-1))^{4n-1}|}{2q(4n-1) \left| (2(4n-1)Ck^{\frac{2m}{4n-1}})^{4n-2} + \dots + (4n^2 - 6n + 1 + 2t_k(4n-1))^{4n-2} \right|}.$$

The numerator is an integer; by the condition $p, q \in P(n)$ it is not equal to zero, therefore ≥ 1 . The denominator is estimated from above:

$$\left| (2(4n-1)Ck^{\frac{2m}{4n-1}})^{4n-2} + \dots \right| \leq Mk^{\frac{4m(2n-1)}{4n-1}}.$$

From here

$$\|z_k\| \geq \frac{1}{Mk^{\frac{4m(2n-1)}{4n-1}}},$$

and by the inequality $|\sin \pi z| \geq 2\|z\|$, we obtain the assertion of the lemma.

Lemma 2 is proved.

Lemma 3. Suppose the following conditions are satisfied: a) Theorem 1 holds; b) C is an algebraic number of degree $d \geq 2$ and $C^{4n-1} \notin \mathbb{Q}$. Then

$$\left| \sin \left(\pi \left\{ Ck^{\frac{2m}{4n-1}} - \frac{4n^2 - 6n + 1}{2(4n-1)} \right\} \right) \right| \geq \frac{M(C, \varepsilon)}{k^{\frac{4m(2n-1)}{4n-1} + 2m + \varepsilon}}, \quad 0 < M = \text{const}, \quad \forall k \in N,$$

for any $\varepsilon > 0$.

Proof. We set $z_k = Ck^{\frac{2m}{4n-1}} - \frac{4n^2 - 6n + 1}{2(4n-1)}$ and choose $t_k \in \mathbb{Z}$ such that $\|z_k\| = |z_k - t_k|$

$$. \text{ Then } 2(4n-1)\|z_k\| = \left| 2(4n-1)Ck^{\frac{2m}{4n-1}} - (4n^2 - 6n + 1 + 2t_k(4n-1)) \right|.$$

Let $m_k = 4n^2 - 6n + 1 + 2t_k(4n-1) \in \mathbb{Z}$ and consider the number $\beta = (2(4n-1))^{4n-1} C^{4n-1}$. Since C is algebraic and $C^{4n-1} \notin \mathbb{Q}$, β is also algebraic and irrational. Applying the identity $a - b = (a^{4n-1} - b^{4n-1}) / (a^{4n-2} + \dots + b^{4n-2})$ to $a = 2(4n-1)Ck^{\frac{2m}{4n-1}}$ and $b = m_k$:

$$\left| 2(4n-1)Ck^{\frac{2m}{4n-1}} - m_k \right| = \frac{k^{2m} \left| \beta - \frac{m_k^{4n-1}}{k^{2m}} \right|}{\left| (2(4n-1))^{4n-2} C^{4n-2} k^{\frac{4m(2n-1)}{4n-1}} + \dots + m_k^{4n-2} \right|}.$$

The denominator does not exceed $Mk^{\frac{4m(2n-1)}{4n-1}}$ (similar to Lemma 2). The fraction m_k^{4n-1} / k^{2m} is a rational approximation to β ; in the irreducible form p_k / q_k , we have $q_k | k^{2m}$, hence $q_k \leq k^{2m}$. By Roth's theorem (applicable because β is irrational and algebraic), for any $\varepsilon' > 0$, we have $M_1 = M_1(\beta, \varepsilon') > 0$:

$$\left| \beta - \frac{p_k}{q_k} \right| \geq \frac{M_1}{q_k^{2+\varepsilon'}} \geq \frac{M_1}{k^{2m(2+\varepsilon')}}.$$

Multiplied by k^{2m} :

$$|\beta k^{2m} - m_k^{4n-1}| \geq \frac{M_1}{k^{2m(1+\varepsilon')}}.$$

Substituting into the estimate above and denoting $\varepsilon = 2m\varepsilon'$:

$$\|z_k\| \geq \frac{M}{k^{\frac{4m(2n-1)}{4n-1} + 2m + \varepsilon}}.$$

By the inequality $|\sin \pi z| \geq 2\|z\|$, we obtain the assertion.

Lemma 3 is proved.

Let's introduce the notation:

$$a_1 = \left[2m + \frac{4m(2n-1)}{4n-1} \right], \quad b_1 = \left\{ 2m + \frac{4m(2n-1)}{4n-1} \right\},$$

$$a_2 = \left[4m + \frac{4m(2n-1)}{4n-1} \right], \quad b_2 = \left\{ 4m + \frac{4m(2n-1)}{4n-1} \right\},$$

where $[x]$ is the integer part, $\{x\}$ is the fractional part of x .

Theorem 3. Let the following conditions be satisfied:

1. Theorem 1 and Lemma 2 hold.
2. $\varphi_j(y), \psi_j(y) \in C^{a_1+1}[0,1]$; $\varphi_j^{(a_1+1)}(y), \psi_j^{(a_1+1)}(y) \in C^\alpha[0,1]$, $b_1 < \alpha \leq 1$, $j = \overline{0, 2n-2}$.
3. $\varphi_j^{(s)}(0) = \varphi_j^{(s)}(1) = \psi_j^{(s)}(0) = \psi_j^{(s)}(1) = 0$, $j = \overline{0, 2n-2}$, $s = 0, 2, 4, \dots, t$,

where

$$t = \begin{cases} a_1 - 1, & a_1 = 2l - 1, l \in \mathbb{N}; \\ a_1, & a_1 = 2l. \end{cases}$$

Then series (2) is a regular solution to problem D .

Proof. Since $\varphi_j^{(a_1+1)} \in C^\alpha[0,1]$, its Fourier coefficients in sines decrease as $O(k^{-\alpha})$ (by the theorem on the rate of decrease of the Fourier coefficients of Holder-continuous functions).

Therefore, $|\varphi_{jk}| \leq \frac{M}{k^{a_1+1+\alpha}}$, $|\psi_{jk}| \leq \frac{M}{k^{a_1+1+\alpha}}$, $M = \text{const} > 0$. For series (2) to define a classical solution, it suffices to establish uniform convergence of the series for u , $\partial u / \partial x$, $\partial u / \partial y$, $\partial^{4n+1} u / \partial x^{4n+1}$, $\partial^{2m} u / \partial y^{2m}$.

The case of $\partial^{2m} u / \partial y^{2m}$. Formally differentiating the series:

$$\frac{\partial^{2m} u}{\partial y^{2m}} = (-1)^m \sum_{k=1}^{\infty} (\pi k)^{2m} X_k(x) \sin \pi k y.$$

Taking Theorem 2 into account, we have

$$\left| \frac{\partial^{2m} u}{\partial y^{2m}} \right| \leq M \sum_{k=1}^{\infty} k^{2m} |X_k(x)| \leq M \sum_{k=1}^{\infty} k^{2m} \frac{\sum_{s=0}^{2n-1} (|\varphi_{sk}| + |\psi_{sk}|)}{Mk^{\frac{4m(2n-1)}{4n-1}}} \leq$$

$$\leq M \sum_{k=1}^{\infty} k^{2m + \frac{4m(2n-1)}{4n-1}} \frac{2n}{k^{a_1+1+\alpha}} = 2nM \sum_{k=1}^{\infty} \frac{1}{k^{1+\alpha-b_1}}.$$

For $\alpha > b_1$, the last series converges, ensuring uniform and absolute convergence.

The case of $\partial^{4n-1}u / \partial x^{4n-1}$. For $x \neq 0$, formally differentiating:

$$\frac{\partial^{4n-1}u}{\partial x^{4n-1}} = \sum_{k=1}^{\infty} X_k^{(4n-1)}(x) \sin \pi k y.$$

From the equation of problem (1): for $x > 0$ $X_k^{(4n-1)}(x) = -(\pi k)^{2m} X_k(x)$, and for $x < 0$ $X_k^{(4n-1)}(x) = (\pi k)^{2m} X_k(x)$, that is $X_k^{(4n-1)}(x) = -\operatorname{sgn}(x)(\pi k)^{2m} X_k(x)$. Therefore,

$$\frac{\partial^{4n-1}u}{\partial x^{4n-1}} = -\operatorname{sgn}(x)(-1)^m \sum_{k=1}^{\infty} (\pi k)^{2m} X_k(x) \sin \pi k y = -\operatorname{sgn}(x) \frac{\partial^{2m}u}{\partial y^{2m}},$$

which is precisely the equation of Problem D . Uniform convergence for fixed $x \neq 0$ follows from the same estimate as for $\partial^{2m}u / \partial y^{2m}$. One-sided limits as $x \rightarrow \pm 0$ exist by construction (from the conjugacy conditions of system (8)). Thus, series (2) is a classical (regular) solution of Problem D .

Theorem 3 is proved.

The next theorem is proved similarly.

Theorem 4. Let the following conditions be satisfied:

1. Theorem 1 and Lemma 3 hold.

2. $\varphi_j(y), \psi_j(y) \in C^{a_2+1}[0,1]$; $\varphi_j^{(a_2+1)}(y), \psi_j^{(a_2+1)}(y) \in C^\alpha[0,1]$, $b_2 < \alpha \leq 1$,

$j = \overline{0, 2n-2}$.

3. $\varphi_j^{(s)}(0) = \varphi_j^{(s)}(1) = \psi_j^{(s)}(0) = \psi_j^{(s)}(1) = 0$, $j = \overline{0, 2n-2}$, $s = 0, 2, 4, \dots, t$,

where

$$t = \begin{cases} a_2 - 1, & a_2 = 2l - 1, l \in \mathbb{N}; \\ a_2, & a_2 = 2l. \end{cases}$$

Then series (2) is a regular solution to problem D .

III. Conclusion. In this paper, we investigate the Dirichlet problem for an odd-order equation $(4n-1)$ with a discontinuous coefficient in a rectangular domain Ω . The solution is constructed by separation of variables in the form of a Fourier series in sines (2), which reduces the problem to a family of odd-order ordinary differential equations (3) with conjugation conditions at the discontinuity line of the coefficient $x = 0$. The discontinuity $\operatorname{sgn}(x)$ results in the characteristic equations for $x > 0$ and $x < 0$ being different ($\lambda^{4n-1} = -\mu_k^{4n-1}$ and $\lambda^{4n-1} = \mu_k^{4n-1}$, respectively); however, the connection of the general solutions through the conjugation conditions at $x = 0$ allows us to construct a unified algebraic system (8).

An analysis of the fundamental determinant Δ of system (8) allowed us to: 1) establish a uniqueness criterion (Theorem 1); 2) calculate the asymptotics of Δ as $k \rightarrow +\infty$ in the form of a compact formula containing the trigonometric factor $\sin(\pi\{Ck^{2m/(4n-1)} - \dots\})$; 3) prove (Lemma 1) that there are only countably many "bad" values of the parameter a that violate the condition $\Delta \neq 0$ for at least one k , and, consequently, "almost all" $a > 0$ ensure uniqueness.

The key difficulty of the problem is the problem of "small denominators": the trigonometric factor can approach zero arbitrarily closely for different k . This difficulty is successfully overcome in two situations covering a wide class of values of the parameter C :

1. Lemma 2: $C^{4n-1} \in \mathbb{Q}$, $C \notin \mathbb{Q}$ is a lower bound for $\sim k^{-4m(2n-1)/(4n-1)}$;

2. Lemma 3: C is algebraic, $C^{4n-1} \notin \mathbb{Q}$ is a lower bound for $\sim k^{-4m(2n-1)/(4n-1)-2m-\varepsilon}$ (Roth's theorem).

The corresponding conditions on the smoothness of the boundary data φ_j, ψ_j are formulated in Theorems 3 and 4. Thus, the necessary smoothness for the existence of a classical solution is determined by the Diophantine properties of C , revealing a deep connection between the theory of partial differential equations and the theory of Diophantine approximations.

Promising directions for further research include the following:

1. Transcendental values of C . For C , a transcendental number, Roth's theorem is inapplicable; here, other tools of the theory of irrationality measures are required (for example, Liouville's theorem for the case of rapidly decreasing approximations) or a transition to probabilistic estimates with respect to the measure on the parameter space.
2. Multidimensional domains. A generalization to the domain $\Omega \subset \mathbb{R}^3$ (an odd-order equation in one variable, elliptic in the other two) opens up possibilities for applications in three-dimensional models of beams, plates, and shells.

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References

- [1]. Dubrovin, B.A., Krichever, I.M., Novikov, S.P. (1985). Integrable systems. I. Dynamical systems-4, Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat. Fund. Napravleniya, 4, 179-277. VINITI, Moscow. [In Russian].
- [2]. Zakharov, V.E., Manakov, S.V., Novikov, S.P., Pitaevskiy, L.P. (1980). Theory of Solitons: the Method of the Inverse Problem. Nauka, Moscow. [In Russian].
- [3]. Ryzhov, O.S., Shefter, G.M. (1965). On the influence of viscosity and thermal conductivity on the structure of compressible flows. Prikladnaya Matematika i Mekhanika, 28(6), 996-1007. [In Russian].
- [4]. Ryzhov, O.S. (1965). Asymptotic flow pattern about rotating bodies in a sonic stream of a viscous heat-conducting gas. Prikladnaya Matematika i Mekhanika, 29(6), 1004-1014. [In Russian].
- [5]. Diesperov, V.N. (1972). On the Green's function of the linearised viscous transonic equation. U.S.S.R. Comput. Math. Math. Phys., 12(5), 225-241.
- [6]. Yusufjon P.A., Rutkauskas S.: On a boundary value problem to third order PDE with multiple characteristics. Nonlinear Anal. Model. Control 16(3), 255-269 (2011).
- [7]. Bendjazia N., Guezane-Lakoud A., Khaldi R. On Third-Order Boundary Value Problems with Multiple Characteristics. Differ Equ Dyn Syst 31, 251-268 (2023). <https://doi.org/10.1007/s12591-019-00507-6>
- [8]. Irgashev, B.Y. Boundary value problem for a degenerate high-order equation. J Elliptic. Parabol. Equ. 11, 1127-1144 (2025). <https://doi.org/10.1007/s41808-025-00367-y>
- [9]. Apakov Yu.P., Ibrokhimov Kh.K. (2025). On solving the second boundary value problem for the Viscous Transonic Equation. Bulletin of the Karaganda University. Mathematics Series, 3(119), 34-45. <https://doi.org/10.31489/2025m3/34-45>
- [10]. Yusufjon P. Apakov. Construction of the Green's Function for One Problem of Rectangular Region. Malaysian Journal of Mathematical Sciences 4(1): 1-16 (2010).
- [11]. Abdrakhmanov A.M. Solvability of a boundary-value problem with an integral boundary condition of the second kind for equations of odd order. Math Notes 88, 151-159 (2010). <https://doi.org/10.1134/S000143461007014X>.
- [12]. Apakov Y.P., Irgashev B.Y. Boundary-Value Problem for a Degenerate High-Odd-Order Equation. Ukr. Math. J. 66, 1475-1490 (2015). <https://doi.org/10.1007/s11253-015-1039-7>
- [13]. Zhuraev, T.D. (1979). Boundary Value Problems for Mixed and Mixed-Composite Type Equations. Fan, Tashkent. [In Russian].
- [14]. Kozhanov I., Potapova S.V. (2018). Boundary value problems for odd order forward-backward-type differential equations with two time variables. Siberian Mathematics Journal, 59(5), 870-884. doi.org/10.1134/S0037446618050117
- [15]. Shubin V.V. Boundary value problems for third-order equations with discontinuous coefficient. Vestnik NGU. Ser. Mat. Mekh. Inform., vol.12, No.1, 126-138 (2012).
- [16]. Irgashev, B.Y. On the Conditions for the Solvability of Boundary-Value Problems for Higher-Order Equations with Discontinuous Coefficients. Math. Notes. 111, 217-229 (2022). <https://doi.org/10.1134/S0001434622010254>
- [17]. Irgashev B.Y. On the solvability of a Dirichlet-type problem for one equation with variable coefficients. Math Meth Appl Sci. 2022; 45: 2685-2700. doi:10.1002/mma.7947

Eksponensial og'irlikli integral chegaralanishlar ostida quvish-qochish differensial o'yini

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Annotatsiya. Maqolada ikki o'yinchi – quvuvchi va qochuvchi ishtirokida, harakatlari integral kvadratik cheklovlar bilan chegaralangan, cheklovlari vaqt bo'yicha eksponensial og'irlikka ega bo'lgan quvish–qochish differensial o'yini qaraladi. Har bir o'yinchining erishuvchanlik to'plami shar ekanligi ko'rsatiladi. Optimal quvish vaqti θ erishuvchanlik sharlarining o'zaro joylashuvidan aniqlanuvchi tenglama orqali beriladi va uning optimalligi quvuvchi hamda qochuvchi uchun strategiyalar qurish yo'li bilan isbotlanadi.

Kalit so'zlar: differensial o'yin, quvish, qochish, integral chegaralanish, erishuvchanlik to'plami, optimal quvish vaqti.

Аннотация. В статье рассматривается дифференциальная игра преследования–убегания двух игроков, преследователя и убегающего. Движения игроков ограничены интегральными квадратичными ограничениями с экспоненциальным весом по времени. Показано, что множество достижимости каждого игрока является шаром. Оптимальное время преследования θ определяется уравнением, которое задаётся взаимным расположением шаров достижимости. Его оптимальность доказывается построением стратегий для преследователя и для убегающего.

Ключевые слова: дифференциальная игра, преследование, убежание, интегральное ограничение, множество достижимости, оптимальное время преследования.

I. Kirish

Yigirmanchi asrning o'rtalarida, Ayzeks tomonidan ziddiyatli muammolarni hal etish bilan shug'ullanuvchi "Differensial o'yinlar" nomli tadqiqot sohasiga asos solindi. Shundan so'ng sohaga doir fundamental natijalar [1, 2, 3, 4, 5] kabi ishlarda chop qilina boshlagan. So'nggi paytlarda differensial o'yinlar nazariyasi sohasida asosiy e'tibor o'yinchilarning boshqaruv funksiyalari geometrik, integral yoki aralash chegaralanishlarga ega bo'lgan holatlarni o'rganishga qaratilgan (masalan, [6, 7, 8, 9, 10, 10]).

Quvish–qochish differensial o'yinlarida o'yinchilarning boshqaruvlari odatda geometrik yoki integral cheklovlar bilan chegaralanadi. Integral chegaralanishda boshqaruvning umumiy "energiyasi" cheklanadi: o'yinchi kuchini vaqt bo'ylab istalgan tarzda taqsimlashi mumkin, ammo sarflangan jami energiya berilgan miqdordan oshmaydi. Ibragimov [11] ishda m ta quvuvchili va bitta qochuvchili sodda harakatli qochish differensial o'yinini o'rgangan. Bu o'yinda quvuvchilarning eng katta tezligi qochuvchining eng katta tezligidan kichik bo'lgan holda qochuvchi cheksiz vaqt davomida qochib yura olishi isbotlangan. Boshqa [12] ishda esa integral chegaralanishga ega bo'lgan quvuvchilar va bitta qochuvchini optimal tutish differensial o'yini qaralgan. O'yinning davomiyligi mahkamlab qo'yilgan va o'yinning to'lov funksiyasi o'yin tugagandan so'ng qochuvchi va quvuvchilar orasidagi minimal masofadir. O'yinning to'lov funksiyasini quyidan baholashga erishilgan.

Ushbu maqolada cheklovlar yanada aniqroq shaklda beriladi: boshqaruv kvadratining $[0, t]$ kesma bo'yicha integrali $\rho^2(1 - e^{-2\kappa t})$ dan oshmaydi. Asosiy maqsad shunday o'yinda quvuvchi quvishni tugatishi garantiyalangan eng kichik vaqtni topish va uning optimalligini isbotlashdir.

II>Masalaning qo'yilishi

$\mathbb{R}^n$ fazoda harakatlanuvchi x quvuvchi va y qochuvchining dinamikasi quyidagi differensial tenglamalar orqali berilgan bo'lsin:

$$\begin{aligned} \dot{x} &= u, & x(0) &= x_0, \\ \dot{y} &= v, & y(0) &= y_0, & x_0 &\neq y_0 \end{aligned} \quad (1)$$

bu yerda $x, y, x_0, y_0 \in \mathbb{R}^n$, u va v mos ravishda x quvuvchi va y qochuvchining boshqaruv parametrlari.

1-ta’rif. Quyidagi tengsizliklarni qanoatlantiruvchi o‘lchovli $u(t) \in \mathbb{R}^n$ va $v(t) \in \mathbb{R}^n, t \geq 0$ funksiyalar

$$\int_0^t |u(s)|^2 ds \leq \rho^2 (1 - e^{-2kt}), \quad (2)$$

$$\int_0^t |v(s)|^2 ds \leq \sigma^2 (1 - e^{-2kt}), \quad (3)$$

mos ravishda x quvuvchi va y qochuvchining joiz boshqaruvlari deb ataladi, bu yerda ρ, σ va k berilgan musbat sonlar.

$\mathbb{U}$ va $\mathbb{V}$ orqali mos ravishda x quvuvchi va y qochuvchining barcha joiz boshqaruvlari to‘plamini belgilab olamiz.

Endi o‘yinchilarning optimal strategiyalari va optimal tutish vaqti uchun ta’riflarni keltiramiz. $H(x, r)$ orqali markazi x nuqtada, radiusi r bo‘lgan sharni belgilaylik.

2-ta’rif. Uzluksiz $U(x_0, y_0, t, v), U: \mathbb{R}^n \times \mathbb{R}^n \times [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ funksiya x quvuvchining strategiyasi deb ataladi, agar ixtiyoriy $v(\cdot) \in \mathbb{V}$ da va $u = U(x_0, y_0, t, v(t))$ da (1) Koshi masalasi yagona $(x(t), y(t))$ yechimga ega bo‘lib, barcha $t \geq 0$ uchun quyidagi

$$\int_0^t |U(x_0, y_0, s, v(s))|^2 ds \leq \rho^2 (1 - e^{-2kt})$$

tengsizlik o‘rinli bo‘lsa.

Boshqacha qilib aytganda, x quvuvchi joriy t vaqtdagi $v(t)$ boshqaruv parametrining qiymatini hamda x_0, y_0 boshlang‘ich holatlar haqidagi ma’lumotlardan foydalanadi.

3-ta’rif. Biror $T(U)$ vaqtda $U = U(x_0, y_0, t, v(t))$ strategiya o‘yinni yakunlaydi deyiladi, agar ixtiyoriy $v(\cdot) \in \mathbb{V}$ hamda biror $\tau \in [0, T(U)]$ uchun $x(\tau) = y(\tau)$ tenglik o‘rinli bo‘lsa, bu yerda $(x(t), y(t))$ (1) Koshi masalasining yechimi hamda $u = U(x_0, y_0, t, v(t))$.

$T(U)$ sonini garantiyalangan tutish vaqti deb ataymiz. Shuni ta’kidlab o‘tish kerakki, $T' \geq T(U)$ shartni qanoatlantiruvchi har qanday T' vaqt xuddi U strategiyaga mos garantiyalangan tutish vaqti bo‘ladi. $T^*(U)$ orqali U strategiyaga mos $T(U)$ sonlar to‘plamining infimumini belgilaylik.

Quvuvchi o‘zlarining U strategiyasini tanlash orqali $T^*(U)$ sonini kichiklashtirishga, qochuvchi esa $v(\cdot) \in \mathbb{V}$ boshqaruvini tanlash orqali $T^*(U)$ sonini kattalashtirishga harakat qiladi. Agar quvuvchining biror U_0 strategiyasi uchun $\inf_{\mathbb{U}} T^*(U) = T^*(U_0)$ o‘rinli bo‘lsa, u holda U_0 quvuvchining optimal strategiyasi va $T^*(U_0)$ soni berilgan o‘yindagi garantiyalangan tutish vaqti deb ataladi.

4-ta’rif. Uzluksiz $V(x_0, y_0, t, x, y), V: \mathbb{R}^n \times \mathbb{R}^n \times [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ funksiyaning qochuvchining strategiyasi deb ataymiz, agar har qanday $u(\cdot) \in \mathbb{U}$ va $v = V(x_0, y_0, t, x, y)$ uchun (1) Koshi masalasi yagona $(x(t), y(t))$ yechimga ega bo‘lib, barcha $t \geq 0$ uchun quyidagi

$$\int_0^t |V(x_0, y_0, s, x, y)|^2 ds \leq \sigma^2 (1 - e^{-2kt})$$

tengsizlik bajarilsa.

5-ta’rif. Agar qochuvchining biror V strategiyasi uchun ixtiyoriy $u(\cdot) \in \mathbb{U}$ va barcha $t \in [0, T(V))$ larda $x(t) \neq y(t)$ tengsizlikka ega bo‘lsak, u holda V strategiya $[0, T(V))$ vaqt intervalida qochishni garantiyalaydi deb ataymiz. $T_*(V)$ orqali V strategiyaga mos $T(V)$ sonlar to‘plamining supremumini belgilaymiz. Shuningdek, $T_*(V)$ sonini V strategiyaga mos garantiyalangan qochish vaqti deb ataymiz.

Qochuvchi V strategiyasini tanlash orqali $T_*(V)$ sonini kattalashtirishga va quvuvchi esa $u(\cdot) \in \mathbb{U}$ boshqaruvini tanlash orqali $T_*(V)$ sonini kichiklashtirishga harakat qilishadi.

6-ta'rif. Agar qochuvchining biror V_0 strategiyasi uchun $\max_V T_*(V) = T_*(V_0)$ o'rinli bo'lsa, u holda V_0 qochuvchining optimal strategiyasi va $T_*(V_0)$ soni berilgan o'yinda garantiyalangan qochish vaqti deb ataladi. Agar $T^*(U_0) = T_*(V_0)$ bo'lsa, u holda bu son (1) o'yinda optimal tutish vaqti deb ataladi.

Endi vaqtning τ momentida quvuvchi erishishi mumkin bo'lgan nuqtalar to'plamini baholaymiz. Integrallash va Koshi- Minkowski tengsizligini qo'llasak:

$$\begin{aligned} |x(\tau) - x_0| &= \left| \int_0^\tau u(s) ds \right| \leq \int_0^\tau |u(s)| ds \\ &\leq \left(\int_0^\tau 1^2 ds \right)^{\frac{1}{2}} \left(\int_0^\tau |u(s)|^2 ds \right)^{\frac{1}{2}} \\ &\leq \rho \sqrt{\tau(1 - e^{-2k\tau})}. \end{aligned}$$

Demak, quvuvchining erishish to'plami markazi x_0 , radiusi $\rho \sqrt{\tau(1 - e^{-2k\tau})}$ bo'lgan $H(x_0, \rho \sqrt{\tau(1 - e^{-2k\tau})})$ shardan iborat bo'ladi. Haqiqatan ham, $\bar{x} \in H(x_0, \rho \sqrt{\tau(1 - e^{-2k\tau})})$ nuqta erishish mumkin bo'lgan nuqta ekanligini ko'rsatamiz. Buning uchun boshqaruvni quyidagicha tanlaymiz:

$$u(t) = \begin{cases} \frac{\bar{x} - x_0}{\tau}, & 0 \leq t \leq \tau, \\ 0, & \tau < t, \end{cases} \text{ bu yerda } |\bar{x} - x_0| \leq \rho \sqrt{\tau(1 - e^{-2k\tau})}.$$

Bu tanlangan boshqaruvning joizligi va $x(\tau) = \bar{x}$ tenglikni qanoatlantirishini tekshiramiz:

$$\begin{aligned} \int_0^\tau |u(t)|^2 dt &= \int_0^\tau \left| \frac{\bar{x} - x_0}{\tau} \right|^2 dt \leq \frac{\rho^2 \tau(1 - e^{-2k\tau})}{\tau^2} \int_0^\tau dt = \rho^2(1 - e^{-2k\tau}) \\ x(\tau) &= x_0 + \int_0^\tau \frac{\bar{x} - x_0}{\tau} dt = x_0 + \bar{x} - x_0 = \bar{x}. \end{aligned}$$

Shunday qilib, quvuvchining erishish to'plami $H(x_0, \rho \sqrt{\tau(1 - e^{-2k\tau})})$ shar hisoblanadi. Xuddi shunday, qochuvchining erishish to'plami $H(y_0, \sigma \sqrt{\tau(1 - e^{-2k\tau})})$ shardan iborat bo'ladi.

Endi optimal tutish vaqti θ ni quyidagi

$$\rho \sqrt{\theta(1 - e^{-2k\theta})} = |y_0 - x_0| + \sigma \sqrt{\theta(1 - e^{-2k\theta})} \quad (4)$$

tenglamadan aniqlaymiz.

Teorema. $\rho \geq \sigma$ sharti bajarilganda, (4) tenglama bilan aniqlangan θ vaqt (1) differensial o'yinda optimal tutish vaqti bo'ladi.

Isbot. Dastlab, θ vaqti kafolatlangan tutish vaqti ekanligini ko'rsatamiz. Quvuvchi quyidagicha

$$u(t) = \begin{cases} \frac{y_0 - x_0}{\theta} + v(t), & 0 \leq t \leq \theta, \\ 0, & \theta < t \end{cases} \quad (5)$$

strategiya bilan harakat qilsin. (5) strategiyaning joizligini ko'rsatishda Koshi-Shwartz tengsizligi va (4) dan foydalanamiz:

$$\left(\int_0^\theta |u(t)|^2 dt \right)^{\frac{1}{2}} \leq \left(\int_0^\theta \left(\frac{|y_0 - x_0|}{\theta} + |v(t)| \right)^2 dt \right)^{\frac{1}{2}}$$

$$\begin{aligned} &\leq \left(\int_0^\theta \left(\frac{|y_0 - x_0|}{\theta} \right)^2 dt \right)^{\frac{1}{2}} + \left(\int_0^\theta |v(t)|^2 dt \right)^{\frac{1}{2}} \leq \frac{|y_0 - x_0|}{\sqrt{\theta}} + \sigma \sqrt{1 - e^{-2k\theta}} \\ &= (\rho - \sigma) \sqrt{1 - e^{-2k\theta}} + \sigma \sqrt{1 - e^{-2k\theta}} = \rho \sqrt{1 - e^{-2k\theta}}. \end{aligned}$$

Quvuvchi (5) strategiya yordamida harakatlenganda, $[0, \theta]$ vaqt oralig'ining oxirgi momentida, ya'ni $t = \theta$ da quvuvchi va qochuvchi trayektoriyalari aynan bir nuqtada uchrashishi, ya'ni $x(\theta) = y(\theta)$ tenglik quyidagicha tekshiriladi:

$$\begin{aligned} x(\theta) &= x^0 + \int_0^\theta \left(\frac{y^0 - x^0}{\theta} + v(t) \right) dt \\ &= x^0 + \left(\frac{y^0 - x^0}{\theta} \right) \int_0^\theta dt + \int_0^\theta v(t) dt = y^0 + \int_0^\theta v(t) dt = y(\theta). \end{aligned}$$

Endi θ vaqtni garantiyalangan qochish vaqti ekanligini isbotlaymiz. Ya'ni qochuvchiga shunday joiz strategiya bilan harakatlanishi kerakki, barcha $0 \leq t < \theta$ vaqtlarda $x(t) \neq y(t)$ tengsizlik bajarilishi lozim. Quyidagicha belgilashlarni kiritamiz:

$$\begin{aligned} \rho^2(t) &= \rho^2(1 - e^{-2\kappa\theta}) - \int_0^t |u(s)|^2 ds, \\ \sigma^2(t) &= \sigma^2(1 - e^{-2\kappa\theta}) - \int_0^t |v(s)|^2 ds \end{aligned}$$

va $e = (y_0 - x_0)/|y_0 - x_0|$ birlik vektorni olamiz. Qochuvchi boshqaruvini quyidagicha tanlaydi:

$$v(t) = \begin{cases} \sigma \sqrt{\frac{1 - e^{-2\kappa\theta}}{\theta}} e, & 0 \leq t \leq \tau \\ 0, & \tau < t \leq \tau + \varepsilon \\ u(t - \varepsilon), & t > \tau + \varepsilon \end{cases} \quad (6)$$

bu yerda $0 < \tau < \theta$ vaqt $\rho(\tau) = \sigma(\tau)$ tenglik birinchi marta bajariladigan vaqt, ε esa quyidagi shartni qanoatlantiradi:

$$0 < \varepsilon < \frac{|y(\tau) - x(\tau)|^2}{4\rho^2}. \quad (6a)$$

Bunday τ vaqt mavjud bo'lmisligi mumkin. U holda qochuvchi boshqaruvi

$$v(t) = \sigma \sqrt{\frac{1 - e^{-2\kappa\theta}}{\theta}} e, \quad 0 \leq t < \theta.$$

Endi quyidagini da'vo qilamiz: agar $v(t) = \sigma \sqrt{\frac{1 - e^{-2\kappa\theta}}{\theta}} e$ bo'lsa, u holda barcha $0 \leq t < \theta$ uchun $x(t) \neq y(t)$ tengsizlik ushbu $\rho(t) \geq \sigma(t)$ bo'lganda bajariladi.

Teskarisini faraz qilamiz: biror t_1 uchun

$$x(t_1) = y(t_1), \quad \rho(t_1) \geq \sigma(t_1), \quad 0 < t_1 < \theta. \quad (7)$$

Bundan va Koshi–Bunyakovskiy tengsizligidan

$$\begin{aligned} x_0 + \int_0^{t_1} u(s) ds &= y(t_1), \quad \Rightarrow \\ |y(t_1) - x_0| &= \left| \int_0^{t_1} u(s) ds \right|, \quad \Rightarrow \end{aligned}$$

$$|y(t_1) - x_0| \leq \int_0^{t_1} |u(s)| ds \leq \sqrt{t_1} \left(\int_0^{t_1} |u(s)|^2 ds \right)^{\frac{1}{2}} \Rightarrow$$

$$\frac{1}{t_1} |y(t_1) - x_0|^2 \leq \left| \int_0^{t_1} |u(s)|^2 ds \right|.$$

$y(t_1) = y_0 + \sigma t_1 \sqrt{\frac{1-e^{-2\kappa\theta}}{\theta}} e$ ekani va e ning ko'rinishidan

$$|y(t_1) - x_0| = |y_0 - x_0| + \sigma t_1 \sqrt{\frac{1 - e^{-2\kappa\theta}}{\theta}}$$

ekani kelib chiqadi. Bundan esa

$$\begin{aligned} \int_0^{t_1} |u(s)|^2 ds - \int_0^{t_1} |v(s)|^2 ds &\geq \frac{1}{t_1} |y(t_1) - x_0|^2 - \frac{\sigma^2 t_1 (1 - e^{-2\kappa\theta})}{\theta} \\ &= \frac{1}{t_1} \left(|y_0 - x_0| + \sigma t_1 \sqrt{\frac{1 - e^{-2\kappa\theta}}{\theta}} \right)^2 - \frac{\sigma^2 t_1 (1 - e^{-2\kappa\theta})}{\theta} \\ &= \frac{1}{t_1} |y_0 - x_0|^2 + 2\sigma \sqrt{\frac{1 - e^{-2\kappa\theta}}{\theta}} |x_0 - y_0| \\ &> \frac{1}{\theta} |y_0 - x_0|^2 + 2\sigma \sqrt{\frac{1 - e^{-2\kappa\theta}}{\theta}} |x_0 - y_0| = \rho^2 (1 - e^{-2\kappa\theta}) - \sigma^2 (1 - e^{-2\kappa\theta}). \end{aligned}$$

Bu yerda $t_1 < \theta$ ekanidan va (4) tenglikdan, ya'ni $|y_0 - x_0| = (\rho - \sigma) \sqrt{\theta(1 - e^{-2\kappa\theta})}$ ekanidan foydalanildi. Demak,

$$\rho^2 (1 - e^{-2\kappa\theta}) - \int_0^{t_1} |u(s)|^2 ds < \sigma^2 (1 - e^{-2\kappa\theta}) - \int_0^{t_1} |v(s)|^2 ds,$$

ya'ni $\rho^2(t_1) < \sigma^2(t_1)$. Bu (7) dagi $\rho(t_1) \geq \sigma(t_1)$ farazga zid. Da'vo isbotlandi.

Boshlang'ich vaqt $t = 0$ da (4) ga ko'ra

$$\rho(0) = \rho \sqrt{1 - e^{-2\kappa\theta}} > \sigma \sqrt{1 - e^{-2\kappa\theta}} = \sigma(0)$$

bo'lgani uchun, agar $\rho(\tau) = \sigma(\tau)$ bo'ladigan birinchi vaqt $\tau \in (0, \theta)$ mavjud bo'lsa, $[0, \tau)$ da

$\rho(t) > \sigma(t)$ bo'ladi. Tanlangan $v(t) = \sigma \sqrt{\frac{1-e^{-2\kappa\theta}}{\theta}} e$ ($0 \leq t \leq \tau$) uchun yuqorida isbotlanganiga ko'ra $x(t) \neq y(t)$, $0 \leq t \leq \tau$.

Endi τ dan keyingi vaqtni ko'ramiz. Agar τ mavjud bo'lmasa, da'voga ko'ra butun $[0, \theta)$ da $x(t) \neq y(t)$ va isbot tugaydi. (6) ga ko'ra $\tau < t \leq \tau + \varepsilon$ da $v(t) = 0$. Bunda $t \in (\tau, \tau + \varepsilon]$ uchun

$$\begin{aligned} |y(t) - x(t)| &= \left| y(\tau) + \int_{\tau}^t 0 ds - x(\tau) - \int_{\tau}^t u(s) ds \right| \geq |y(\tau) - x(\tau)| - \left| \int_{\tau}^t u(s) ds \right| \\ &\geq |y(\tau) - x(\tau)| - \rho \sqrt{\varepsilon} > \frac{1}{2} |y(\tau) - x(\tau)| > 0, \end{aligned}$$

bu yerda $|\int_{\tau}^t u(s)ds| \leq \sqrt{t-\tau} \rho \leq \rho\sqrt{\varepsilon}$ va (6a) dan $\rho\sqrt{\varepsilon} < \frac{1}{2}|y(\tau) - x(\tau)|$ ekanidan foydalanildi. Demak, $x(t) \neq y(t)$, $\tau < t \leq \tau + \varepsilon$.

Nihoyat, $x(t) \neq y(t)$ ekanini $\tau + \varepsilon < t < \theta$ uchun isbotlaymiz. $t > \tau + \varepsilon$ da

$$|y(t) - x(t)| = \left| y(\tau) + \int_{\tau}^t v(s)ds - x(\tau) - \int_{\tau}^t u(s)ds \right|. \quad (8)$$

Yuqoridagi (6) ga ko'ra $v(s) = 0$ ($\tau < s \leq \tau + \varepsilon$) va $v(s) = u(s - \varepsilon)$ ($s > \tau + \varepsilon$) bo'lgani uchun

$$\int_{\tau}^t v(s)ds = \int_{\tau+\varepsilon}^t u(s-\varepsilon)ds = \int_{\tau}^{t-\varepsilon} u(s)ds.$$

Shu sababli (8) ga ko'ra

$$\begin{aligned} |y(t) - x(t)| &= \left| y(\tau) - x(\tau) + \int_{\tau}^{t-\varepsilon} u(s)ds - \int_{\tau}^t u(s)ds \right| \\ &= \left| y(\tau) - x(\tau) - \int_{t-\varepsilon}^t u(s)ds \right| \\ &\geq |y(\tau) - x(\tau)| - \int_{t-\varepsilon}^t |u(s)|ds. \end{aligned} \quad (9)$$

Koshi–Bunyakovskiy tengsizligi va (6a) ga ko'ra

$$\begin{aligned} \int_{t-\varepsilon}^t |u(s)|ds &\leq \left(\int_{t-\varepsilon}^t 1^2 ds \right)^{\frac{1}{2}} \left(\int_{t-\varepsilon}^t |u(s)|^2 ds \right)^{\frac{1}{2}} \leq \sqrt{\varepsilon} \rho \\ &< \frac{1}{2} |y(\tau) - x(\tau)|. \end{aligned}$$

Demak, $t > \tau + \varepsilon$ uchun (9) dan

$$|y(t) - x(t)| > |y(\tau) - x(\tau)| - \frac{1}{2} |y(\tau) - x(\tau)| = \frac{1}{2} |y(\tau) - x(\tau)| > 0,$$

ya'ni $x(t) \neq y(t)$, $t > \tau + \varepsilon$.

Shunday qilib, agar biror $0 < \tau < \theta$ uchun $\rho(\tau) = \sigma(\tau)$ bo'lsa, barcha $t \geq \tau$ da $x(t) \neq y(t)$, xususan $x(t) \neq y(t)$, $\tau \leq t < \theta$. Bundan $[0, \theta)$ da qochish mumkinligi, demak θ ning optimal quvish vaqti ekani kelib chiqadi. Teorema isbotlandi.

III. Xulosa

Maqolada integral cheklovlari eksponensial og'irlikli quvish–qochish o'yini tahlil qilindi. Quvuvchi va qochuvchining erishuvchanlik to'plamlari markazlari boshlang'ich holatlarda, radiuslari mos ravishda $\rho\sqrt{\tau(1 - e^{-2\kappa\tau})}$ va $\sigma\sqrt{\tau(1 - e^{-2\kappa\tau})}$ bo'lgan sharlardan iborat ekani ko'rsatildi. Optimal quvish vaqti θ bu sharlarning ichki urinish sharti (4) orqali aniqlanadi. Quvuvchining (5) strategiyasi θ vaqtda quvishni tugatishini kafolatlaydi, qochuvchining (6) strategiyasi esa $[0, \theta)$ da uchrashuvdan qochishni ta'minlaydi.

Foydalanilgan adabiyotlar

1. Isaacs R. Differential games/ John Wiley & Sons, New York, NY, USA. (1965)
2. Krasovskii N.N. Subbotin A.I. Game-theoretical control problems/ Springer, New York (1988)
3. Petrosyan L.A. Differential games of pursuit/ World Scientific, Singapore (1993)
4. Pshenichnii B.N. Simple pursuit by several objects/ Cybernetics and System Analysis. – 12(3): 484-485. <https://doi.org/10.1007/BF01070036> (1976)
5. Ibragimov G.I. Satimov N. A multi player pursuit differential game on a convex set with *Scientific Bulletin. Physical and Mathematical Research* Vol 8. Iss 3. 2026

- integral constraints/ Abstract and Applied Analysis. <https://doi:10.1155/2012/460171> (2012)
6. Salimi M. Ferrara, M. Differential game of optimal pursuit of one evader by many pursuers/ International Journal of Game Theory. <https://doi.org/10.1007/s00182-018-0638-6> (2018)
7. Samatov B. Problems of group pursuit with integral constraints on controls of the players/ Cybernetics and System Analysis. 49(5), 756-767, (2013)
8. Samatov B. Problems of group pursuit with integral constraints on controls of the players/ Cybernetics and System Analysis. 49(6), 907-921, (2013)
9. Ibragimov G.I. An n-person Differential game with integral constraints on control of the players/ Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika 6, 48-52, (2004)
10. Satimov N. Rikhsiev B. Khamdamov A. On pursuit problem for linear differential and discrete n-person games with integral constraints/ Matematicheskii Sbornik. 46(4), 459-471, (1993)
11. Ibragimov G.I. Evasion differential game of one evader and many slow pursuers/ Dynamic games and applications. <https://doi.org/10.1007/s13235-023-00501-2> (2023)
12. Kuchkarov A.Sh. Ibragimov G.I. Khakestari M. On a linear differential game of optimal approach of many pursuers with one evader/ Journal of Dynamical and Control Systems. 19(1): 1-15, <https://doi.org/10.1007/s10883-013-9161-z> (2013)

Boshqaruvlarga Gronuoll tipidagi cheklovlar qo'yilgan kasr tartibli oddiy harakat differensial o'yinida qochish masalasi

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Annotatsiya. Maqolada bitta quvuvchi va bitta qochuvchi ishtirokidagi oddiy harakat differensial o'yini o'rganiladi. O'yinchilarning harakati Kaputo ma'nosidagi kasr tartibli differensial tenglamalar bilan tavsiflanadi, boshqaruvlarga esa Gronuoll tipidagi cheklovlar qo'yiladi. Qochuvchining imkoniyati quvuvchidan kam bo'lmagan holda qochish masalasi to'liq yechiladi. Qochuvchining quvuvchidan bevosita uzoqlashuvchi strategiyasi qochishni kafolatlashi va o'yinchilar orasidagi masofani aniq ko'rsatilgan miqdordan kam bo'lmashini ta'minlashi isbotlanadi. Quvuvchining mutanosib quvish strategiyasi masofani xuddi shu miqdordan oshirmasligi ko'rsatiladi. Shunday qilib, bu miqdor kafolatlangan masofa o'yinining qiymati bo'lib, ikkala strategiya optimaldir. Qochish sharti kasr tartibga va Gronuoll parametriga bog'liq emasligi aniqlanadi. Avval olingan quvish natijasi bilan birgalikda o'yinning to'liq manzarasi hosil qilinadi.

Kalit so'zlar: differensial o'yin, qochish, quvuvchi, qochuvchi, Kaputo kasr hosilasi, Gronuoll tipidagi cheklov, Mittag-Leffler funksiyasi, kafolatlangan masofa, optimal strategiya.

Аннотация. Изучается дифференциальная игра простого движения одного преследователя и одного убегающего, динамика которых описывается дробными дифференциальными уравнениями Капуто, а управления подчинены ограничениям типа Гронуолла. В случае, когда ресурс убегающего не меньше ресурса преследователя, задача убегающего решается полностью. Доказано, что стратегия убегающего, направленная от преследователя, гарантирует убежание и сохраняет расстояние между игроками не меньшим явно указанной величины, а стратегия пропорционального преследования сохраняет расстояние не большим той же величины. Тем самым эта величина является значением игры гарантированного расстояния, а обе стратегии оптимальны. Показано, что условие убегающего не зависит от дробного порядка и параметра Гронуолла. Вместе с ранее полученным результатом о преследовании получена полная картина игры.

Ключевые слова: дифференциальная игра, убежание, преследователь, убегающий, дробная производная Капуто, ограничение типа Гронуолла, функция Миттаг-Леффлера, гарантированное расстояние, оптимальная стратегия.

Abstract. We study a simple-motion differential game of one pursuer and one evader whose dynamics are described by Caputo fractional differential equations and whose controls are subject to Grönwall-type constraints. When the evader's resource is not smaller than that of the pursuer, the evasion problem is solved completely. It is proved that the strategy of the evader directed away from the pursuer guarantees evasion and keeps the distance between the players not less than an explicitly given quantity, while the proportional pursuit strategy of the pursuer keeps the distance not greater than the same quantity. Hence this quantity is the value of the guaranteed-distance game and both strategies are optimal. The evasion condition is shown to be independent of the fractional order and of the Grönwall parameter. Together with the earlier pursuit result, a complete picture of the game is obtained.

Keywords: differential game, evasion, pursuer, evader, Caputo fractional derivative, Grönwall-type constraint, Mittag-Leffler function, guaranteed distance, optimal strategy.

I. Kirish.

Quvish–qochish differensial o'yinlari nazariyasi R. Ayzeks [1] va L. S. Pontryagin [2] ishlaridan boshlangan bo'lib, hozirgi kunda matematik boshqaruv nazariyasining faol rivojlanayotgan yo'nalishlaridan biridir [3]. Bu nazariyaning eng sodda va shu bilan birga fundamental modeli — oddiy harakat o'yini. Unda har bir o'yinchi o'z tezligini bevosita boshqaradi va boshqaruvga qo'yilgan cheklov o'yinchining “imkoniyati”ni belgilaydi. Geometrik cheklovlar holda, ya'ni tezliklar $|u| \leq \rho$ va $|v| \leq \sigma$ o'zgarimas sonlar bilan chegaralanganda, bitta quvuvchi va bitta qochuvchi o'yini to'liq yechilgan: quvuvchi tezroq bo'lsa ($\rho > \sigma$), parallel yaqinlashish

strategiyasi (II-strategiya, [4]) chekli vaqtda tutishni ta'minlaydi; aks holda ($\sigma \geq \rho$) qochuvchi quvuvchidan bevosita uzoqlashib, hech qachon tutilmaydi.

So'nggi yillarda boshqaruvlarga Gronuoll tipidagi cheklovlar qo'yilgan o'yinlar faol o'rganilmoqda [5–10]. Bunday cheklov T. H. Gronuollning klassik integral tengsizligiga [11] asoslanadi va boshqaruvning joriy qiymatini uning o'tmishdagi qiymatlari bilan bog'laydi: boshqaruv qancha ko'p sarflangan bo'lsa, keyingi paytda shuncha katta boshqaruvga ruxsat beriladi. Natijada o'yinchi vaqt o'tishi bilan eksponensial sur'atda tezlanish imkoniga ega bo'ladi. B. T. Samatov, G. I. Ibragimov va I. V. Xodjiboyeva [5] bunday cheklovli oddiy harakat o'yinini bitta

quvuvchi va bitta qochuvchi uchun yechib, II-strategiyaning optimalligini isbotladilar. Keyinchalik ikkita quvuvchi holi [7], ko'p quvuvchi holi [8], belgilangan davomiylikdagi o'yin [9] va "hayot chizig'i" o'yini [10] o'rganildi.

Boshqa tomondan, xotira va irsiyat effektlariga ega jarayonlarni modellashtirishda kasr tartibli hosilalar [12–14] keng qo'llanilmoqda. Kasr tartibli dinamikali differensial o'yinlar A. A. Chikriy va I. I. Matichin [15] ishlaridan boshlab o'rganilmoqda. Kaputo ma'nosidagi kasr hosilali oddiy harakat o'yinida bitta quvuvchi holida optimal quvish vaqti geometrik cheklovlar uchun [16] da, Gronuoll tipidagi cheklovlar uchun esa [17] da topilgan. Bu ikkala ishda ham quvuvchi tezroq deb faraz qilingan, ya'ni faqat quvish masalasi qaralgan.

Ushbu maqolada [17] ning mantiqiy davomi sifatida qarama-qarshi hol — qochuvchining imkoniyati quvuvchinikidan kam bo'lmagan hol — qaraladi va qochish masalasi to'liq yechiladi. Maqolaning asosiy natijalari quyidagilardan iborat.

– Qochuvchi uchun quvuvchidan bevosita uzoqlashuvchi strategiya quriladi va u qochishni kafolatlashi, bunda o'yinchilar orasidagi masofa aniq ko'rsatilgan $D(t)$ miqdordan kam bo'lmashligi isbotlanadi (Teorema 1).

– Quvuvchi uchun mutanosib quvish strategiyasi quriladi va u masofani xuddi shu $D(t)$ miqdordan oshirmasligi isbotlanadi. Demak, $D(t)$ — kafolatlangan masofa o'yinining qiymati, ikkala strategiya esa optimaldir (Teorema 2).

– Qochish sharti kasr tartibga ham, Gronuoll parametriga ham bog'liq emasligi aniqlanadi (Natija 1) va [17] bilan birgalikda o'yinning to'liq manzarasi olinadi (Natija 2).

Maqola quyidagicha tuzilgan. 2-bo'limda masala qo'yiladi va zarur ta'riflar keltiriladi. 3-bo'limda Gronuoll bahosi va "samarali vaqt" funksiyasi o'rganiladi. 4-bo'limda o'yinchilarning strategiyalari quriladi va ularning xossalari isbotlanadi. 5-bo'limda asosiy teoremlar bayon qilinadi. 6-bo'limda natijalar muhokama qilinadi va sonli misol keltiriladi.

II. Material va usullar.

2.1. Masalaning qo'yilishi

Kasr tartibli hosila va integral

Butun maqola davomida R^n ($n \geq 2$) — $\langle \cdot, \cdot \rangle$ skalyar ko'paytmali va $|\cdot|$ Evklid normal fazo, $\Gamma(\cdot)$ — Eyler gamma-funksiyasi.

$0 < \alpha \leq 1$ bo'lsin. f funksiyaning α tartibli Riman–Liuvill kasr integrali deb

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds \quad (1)$$

ifodaga, $0 < \alpha < 1$ da Kaputo kasr hosilasi deb

$${}^C D_{0+}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds \quad (2)$$

ifodaga aytiladi [12]. $\alpha = 1$ da (1) oddiy integral, (2) esa oddiy hosila deb tushuniladi. Bizga kerak bo'ladigan asosiy fakt [12]: ${}^C D_{0+}^\alpha f = g$, $f(0) = f_0$ Koshi masalasining yechimi

$$f(t) = f_0 + I^\alpha g(t)$$

ko'rinishda bo'ladi. Ya'ni kasr tenglamani yechish — o'ng tomonning kasr integralini olishdir.

O'yinchilarning harakati va cheklovlar

R^n fazoda quvuvchi P va qochuvchi E harakatlanadi. Ularning holatlari $x(t)$ va $y(t)$ quyidagi Kaputo kasr tartibli tenglamalarni qanoatlantiradi:

$$\begin{aligned} {}^C D_{0+}^\alpha x(t) &= u(t), & x(0) &= x_0, \\ {}^C D_{0+}^\alpha y(t) &= v(t), & y(0) &= y_0, \end{aligned} \quad (3)$$

bu yerda $u(t) \in R^n$ — quvuvchining, $v(t) \in R^n$ — qochuvchining boshqaruv parametri, $x_0 \neq y_0$. Boshqaruvlarga Gronuoll tipidagi cheklovlar qo'yilgan:

$$\begin{aligned} |u(t)|^2 &\leq \rho^2 + k \int_0^t |u(s)|^2 ds, \\ |v(t)|^2 &\leq \sigma^2 + k \int_0^t |v(s)|^2 ds \end{aligned} \quad (4)$$

deyarli barcha $t \geq 0$ uchun; bu yerda $\rho > 0$, $\sigma > 0$, $k \geq 0$ — berilgan sonlar. ρ va σ mos ravishda quvuvchi va qochuvchining boshlang'ich imkoniyatini, k esa tezlanish sur'atini belgilaydi; $k = 0$ da (4) odatdagi geometrik cheklovlarga aylanadi. Butun maqola davomida

$$\sigma \geq \rho$$

deb faraz qilinadi, ya'ni qochuvchining imkoniyati quvuvchinikidan kam emas.

Yuqorida aytilgan faktga ko'ra (3) tenglamalarning yechimlari

$$x(t) = x_0 + I^\alpha u(t), \quad y(t) = y_0 + I^\alpha v(t) \quad (5)$$

ko'rinishda yoziladi.

O'yinchilarning boshlang'ich holatlari bilan bog'liq ikki belgilash kiritamiz:

$$d = |y_0 - x_0|, \quad \xi_0 = \frac{x_0 - y_0}{|x_0 - y_0|}.$$

Bu yerda $d > 0$ — boshlang'ich masofa, ξ_0 — qochuvchidan quvuvchi tomonga yo'nalgan birlik vektor. Ravshanki, $y_0 = x_0 - d \xi_0$.

Ta'riflar

Ta'rif 1. $[0, \infty)$ da o'lchovli, har bir chekli kesmada kvadrati bilan integrallanuvchi va (4) dagi birinchi (mos ravishda ikkinchi) shartni

qanoatlantiruvchi $u(\cdot)$ (mos ravishda $v(\cdot)$) funksiya quvuvchining (mos ravishda qochuvchining) ruxsat etilgan boshqaruvi deyiladi.

Ta'rif 2. Agar biror $\tau \geq 0$ paytda $x(\tau) = y(\tau)$ bo'lsa, o'yinda *tutish sodir bo'ldi* deyiladi. Agar barcha $t \geq 0$ uchun $x(t) \neq y(t)$ bo'lsa, *qochish sodir bo'ldi* deyiladi.

Ta'rif 3. Qochuvchining ruxsat etilgan $v(\cdot)$ boshqaruvi *qochishni kafolatlaydi* deyiladi, agar quvuvchining istalgan ruxsat etilgan $u(\cdot)$ boshqaruvida qochish sodir bo'lsa. Shunday boshqaruv mavjud bo'lsa, *qochish masalasi yechiladi* deyiladi.

Ta'kidlaymizki, qochuvchi uchun dasturli, ya'ni oldindan belgilangan boshqaruv qaralmoqda. Bunday boshqaruv qochishni kafolatlasa, u qochuvchining istalgan axborotli strategiyalari sinfida ham qochishni kafolatlaydi. Quvuvchi esa istalgan axborotdan, jumladan qochuvchi boshqaruvining joriy qiymatidan foydalanishi mumkin. Buni rasmiylashtiramiz.

Ta'rif 4. Quvuvchining kontrstrategiyasi deb shunday $u = u(t, v)$ funksiyaga aytiladiki, qochuvchining har qanday ruxsat etilgan $v(\cdot)$ boshqaruvi uchun $u(t) = u(t, v(t))$ quvuvchining ruxsat etilgan boshqaruvi bo'lsin.

Maqolaning maqsadi — $\sigma \geq \rho$ da qochish masalasini yechish va bunda o'yinchilar orasidagi masofani ikkala tomon uchun ham aniq baholash.

2.2. Gronuoll bahosi va samarali vaqt

Gronuoll tipidagi cheklovning asosiy xususiyati shundaki, u boshqaruv uchun oshkor, vaqtga bog'liq chegara beradi.

Lemma 1 (Gronuoll bahosi). $u(\cdot)$ va $v(\cdot)$ ruxsat etilgan boshqaruvlar bo'lsin. U holda deyarli barcha $t \geq 0$ uchun

$$|u(t)| \leq \rho e^{kt/2}, \quad |v(t)| \leq \sigma e^{kt/2}. \quad (6)$$

Isbot. Qochuvchi uchun isbotlaymiz, quvuvchi uchun xuddi shunday. $\varphi(t) = |v(t)|^2$ va $\psi(t) = \sigma^2 + k \int_0^t \varphi(s) ds$ deb olamiz. φ lokal integrallanuvchi bo'lgani uchun ψ absolyut uzluksiz, $\psi(0) = \sigma^2$ va deyarli barcha t uchun $\psi'(t) = k\varphi(t)$. (4) ga ko'ra $\varphi(t) \leq \psi(t)$, demak $\psi'(t) \leq k\psi(t)$. Bundan

$$\frac{d}{dt}(e^{-kt}\psi(t)) = e^{-kt}(\psi'(t) - k\psi(t)) \leq 0,$$

ya'ni $e^{-kt}\psi(t) \leq \psi(0) = \sigma^2$. Demak $|v(t)|^2 = \varphi(t) \leq \psi(t) \leq \sigma^2 e^{kt}$. □

Shunday qilib, Gronuoll cheklovli o'yinchi t paytda ko'pi bilan $\sigma e^{kt/2}$ (mos ravishda $\rho e^{kt/2}$) "tezlik" bilan harakatlana oladi. Kasr dinamikada esa holat tezlikning oddiy integrali emas, kasr integrali orqali aniqlanadi. Shu sababli quyidagi funksiya markaziy o'rin tutadi:

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{ks/2} ds, \quad t \geq 0, \quad (7)$$

ya'ni $\Phi_\alpha = I^\alpha(e^{ks/2})$. $\Phi_\alpha(t)$ — birlik imkoniyatli ($\sigma = 1$) o'yinchining to'la tezlik bilan bir yo'nalishda harakatlanganda t paytgacha bosib o'tgan masofasi. Uni *samarali vaqt* deb ataymiz: klassik holda ($\alpha = 1$, $k = 0$) u t ning o'ziga teng.

Lemma 2. Φ_α funksiya quyidagi xossalarga ega:

(a) $\Phi_\alpha(t) = t^\alpha E_{1,\alpha+1}(kt/2)$, bu yerda $E_{1,\beta}(z) = \sum_{j \geq 0} z^j / \Gamma(j + \beta)$ — ikki parametrlı Mittag-Leffler funksiyasi [14];

(b) Φ_α funksiya $[0, \infty)$ da uzluksiz va qat'iy o'suvchi, $\Phi_\alpha(0) = 0$, $\Phi_\alpha(t) \rightarrow \infty$ ($t \rightarrow \infty$);

(c) barcha $t \geq 0$ uchun

$$\frac{t^\alpha}{\Gamma(\alpha+1)} \leq \Phi_\alpha(t) \leq \frac{t^\alpha}{\Gamma(\alpha+1)} e^{kt/2}; \quad (8)$$

(d) $k = 0$ da $\Phi_\alpha(t) = t^\alpha / \Gamma(\alpha+1)$; $\alpha = 1$, $k > 0$ da $\Phi_1(t) = 2/k(e^{kt/2} - 1)$; $\alpha = 1$, $k = 0$ da $\Phi_1(t) = t$.

Isbot. (a) $e^{ks/2} = \sum_{j \geq 0} (k/2)^j s^j / j!$ yoyilmani (7) ga qo'yib hadma-had integrallaymiz va $I^\alpha s^j = j! t^{j+\alpha} / \Gamma(j + \alpha + 1)$ formulasidan foydalanamiz:

$$\begin{aligned} \Phi_\alpha(t) &= \sum_{j \geq 0} \frac{(k/2)^j}{j!} \cdot \frac{j! t^{j+\alpha}}{\Gamma(j + \alpha + 1)} \\ &= t^\alpha \sum_{j \geq 0} \frac{(kt/2)^j}{\Gamma(j + \alpha + 1)}. \end{aligned}$$

(Bu yerda $I^\alpha s^j$ formulasi $s = t\tau$ almashtirish va Eyler beta-integrali $B(j+1, \alpha) = j! \Gamma(\alpha) / \Gamma(j + \alpha + 1)$ orqali olinadi.) (b) (7) da $\tau = t - s$ almashtirsak,

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \tau^{\alpha-1} e^{k(t-\tau)/2} d\tau.$$

t o'sganda integral ostidagi manfiy masofa funksiya kamaymaydi va integrallash oralig'i kengayadi; demak Φ_α qat'iy o'suvchi. Uzluksizlik (a) dan, $\Phi_\alpha(t) \rightarrow \infty$ esa (c) dagi quyi bahodan kelib chiqadi. (c) $0 \leq s \leq t$ da $1 \leq e^{ks/2} \leq e^{kt/2}$, va $I^\alpha 1 = t^\alpha / \Gamma(\alpha + 1)$. (d) bevosita hisoblash bilan tekshiriladi. □

Lemma 2 dan o'yinchining erishuvchanlik sohasi haqidagi muhim xulosa kelib chiqadi.

Lemma 3 (erishuvchanlik sohasi). $u(\cdot)$ quvuvchining, $v(\cdot)$ qochuvchining istalgan ruxsat etilgan boshqaruvlari bo'lsin. U holda barcha $t \geq 0$ uchun

$$|x(t) - x_0| \leq \rho \Phi_\alpha(t), \quad |y(t) - y_0| \leq \sigma \Phi_\alpha(t). \quad (9)$$

Isbot. (5), (6) va (7) dan

$$\begin{aligned}
|x(t) - x_0| &= |I^\alpha u(t)| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |u(s)| ds \\
&\leq \frac{\rho}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{ks/2} ds \\
&= \rho \Phi_\alpha(t).
\end{aligned}$$

Qochuvchi uchun xuddi shunday. $\square$

Ya'ni t paytda quvuvchi markazi x_0 , radiusi $\rho\Phi_\alpha(t)$ bo'lgan shardan, qochuvchi esa markazi y_0 , radiusi $\sigma\Phi_\alpha(t)$ bo'lgan shardan tashqariga chiqa olmaydi.

2.3. O'yinchilarning strategiyalari

Qochuvchining uzoqlashish strategiyasi

Qochuvchi uchun eng tabiiy strategiya — quvuvchidan qarama-qarshi tomonga, to'la imkoniyat bilan harakatlanish.

Ta'rif 5. Qochuvchining

$$v^*(t) = -\sigma e^{kt/2} \xi_0, \quad t \geq 0, \quad (10)$$

boshqaruvi uzoqlashish strategiyasi deyiladi.

Lemma 4. Uzoqlashish strategiyasi ruxsat etilgan, bunda (4) dagi ikkinchi shart tenglik bilan bajariladi. Unga mos qochuvchi trayektoriyasi

$$y(t) = y_0 - \sigma \Phi_\alpha(t) \xi_0, \quad t \geq 0, \quad (11)$$

ya'ni qochuvchi y_0 nuqtadan quvuvchiga qarama-qarshi yo'nalgan nurda harakatlanadi va t paytda y_0 dan $\sigma\Phi_\alpha(t)$ masofada bo'ladi.

Isbot. $|v^*(t)|^2 = \sigma^2 e^{kt}$ va $\sigma^2 + k \int_0^t \sigma^2 e^{ks} ds = \sigma^2 e^{kt}$, demak (4) tenglik bilan bajariladi; o'lchovlilik va lokal integrallanuvchanlik ravshan. (11) esa (5) va (7) dan bevosita kelib chiqadi, chunki ξ_0 o'zgarmas vektor va I^α chiziqli. $\square$

Quvuvchining mutanosib quvish strategiyasi

Quvuvchi uchun esa qochuvchining boshqaruvini "nusxalash" — uni ρ/σ nisbatda takrorlash — tabiiy kontrstrategiyadir.

Ta'rif 6. Quvuvchining

$$u^*(t, v) = \frac{\rho}{\sigma} v \quad (12)$$

kontrstrategiyasi mutanosib quvish strategiyasi deyiladi.

Lemma 5. Mutanosib quvish strategiyasi ruxsat etilgan kontrstrategiya: qochuvchining istalgan ruxsat etilgan $v(\cdot)$ boshqaruvi uchun $u(t) = \rho/\sigma v(t)$ (4) dagi birinchi shartni qanoatlantiradi. Bunda o'yinchilar orasidagi masofa vektori

$$y(t) - x(t) = -d\xi_0 + \left(1 - \frac{\rho}{\sigma}\right)(y(t) - y_0). \quad (13)$$

Isbot. (4) dagi ikkinchi shartni ρ^2/σ^2 ga ko'paytiramiz:

$$\begin{aligned}
|u(t)|^2 &= \frac{\rho^2}{\sigma^2} |v(t)|^2 \\
&\leq \frac{\rho^2}{\sigma^2} \left(\sigma^2 + k \int_0^t |v(s)|^2 ds \right) \\
&= \rho^2 + k \int_0^t \frac{\rho^2}{\sigma^2} |v(s)|^2 ds \\
&= \rho^2 + k \int_0^t |u(s)|^2 ds.
\end{aligned}$$

Bu aynan (4) dagi birinchi shart. O'lchovlilik va lokal integrallanuvchanlik v dan meros bo'ladi. (13) ni olish uchun (5) dan foydalanamiz: I^α chiziqli bo'lgani uchun

$$\begin{aligned}
y(t) - x(t) &= y_0 - x_0 + I^\alpha(v - u)(t) \\
&= y_0 - x_0 + \left(1 - \frac{\rho}{\sigma}\right) I^\alpha v(t) \\
&= -d\xi_0 + \left(1 - \frac{\rho}{\sigma}\right)(y(t) - y_0).
\end{aligned}$$

(13) ning ma'nosi: mutanosib quvishda masofa vektori boshlang'ich masofa vektori bilan qochuvchi siljishining $(1 - \rho/\sigma)$ ulushidan tashkil topadi. Qochuvchi qancha ko'p siljisa, quvuvchi ham shuncha ko'p siljiydi, faqat ρ/σ marta kamroq.

III. Natijalar va muhokama.

3.1. Asosiy natijalar

Quyidagi funksiyani kiritamiz:

$$D(t) = d + (\sigma - \rho) \Phi_\alpha(t), \quad t \geq 0. \quad (14)$$

Bu — boshlang'ich masofa va imkoniyatlar farqining samarali vaqtga ko'paytmasi yig'indisi. Asosiy natijalar $D(t)$ ning o'yindagi rolini ikki tomondan ochib beradi.

Teorema 1 (qochish). $\sigma \geq \rho$ bo'lsin. U holda uzoqlashish strategiyasi (10) qochishni kafolatlaydi, bunda quvuvchining istalgan ruxsat etilgan $u(\cdot)$ boshqaruvi uchun

$$|y(t) - x(t)| \geq D(t), \quad t \geq 0. \quad (15)$$

Xususan, $\sigma > \rho$ da o'yinchilar orasidagi masofa cheksiz o'sadi, $\sigma = \rho$ da esa u boshlang'ich masofa d dan hech qachon kam bo'lmaydi.

Isbot. Qochuvchi (10) strategiyani qo'llasin; Lemma 4 ga ko'ra u ruxsat etilgan va (11) o'rinli. $y_0 = x_0 - d\xi_0$ ni (11) ga qo'ysak,

$$y(t) - x_0 = -(d + \sigma\Phi_\alpha(t))\xi_0,$$

demak

$$|y(t) - x_0| = d + \sigma\Phi_\alpha(t).$$

Boshqacha aytganda, qochuvchi quvuvchining boshlang'ich holatidan $d + \sigma\Phi_\alpha(t)$ masofaga uzoqlashadi. Quvuvchi esa Lemma 3 ga ko'ra o'z boshlang'ich holatidan $\rho\Phi_\alpha(t)$ dan ortiq uzoqlasha olmaydi. Uchburchak tengsizligidan

$$|y(t) - x(t)| \geq |y(t) - x_0| - |x(t) - x_0| \\ \geq d + \sigma \Phi_\alpha(t) - \rho \Phi_\alpha(t) = D(t).$$

$\sigma \geq \rho$ bo'lgani uchun $D(t) \geq d > 0$, demak $x(t) \neq y(t)$ barcha $t \geq 0$ uchun — qochish sodir bo'ladi. Oxirgi tasdiq Lemma 2(b) dan kelib chiqadi: $\sigma > \rho$ da $D(t) \rightarrow \infty$. $\square$

Teorema 1 qochuvchi $D(t)$ masofani kafolatlay olishini ko'rsatadi. Tabiiy savol: u bundan ko'prog'ini kafolatlay oladimi? Javob salbiy.

Teorema 2 (kafolatlangan masofa o'yinining qiymati). $\sigma \geq \rho$ bo'lsin. U holda har bir $t \geq 0$ uchun

$$\sup_v \inf_u |y(t) - x(t)| = \inf_u \sup_v |y(t) - x(t)| \\ = D(t), \quad (16)$$

bu yerda $\sup$ qochuvchining ruxsat etilgan boshqaruvlari bo'yicha, $\inf$ esa chap tomonda quvuvchining ruxsat etilgan boshqaruvlari, o'ng tomonda kontrstrategiyalari bo'yicha olinadi. Uzoqlashish strategiyasi (10) va mutanosib quvish strategiyasi (12) optimaldir.

Isbot. Quyi baho. Teorema 1 ga ko'ra qochuvchi (10) ni qo'llaganda quvuvchining istalgan boshqaruvi uchun $|y(t) - x(t)| \geq D(t)$. Demak (16) dagi chap tomon $D(t)$ dan kam emas.

Yuqori baho. Quvuvchi (12) ni qo'llasin; Lemma 5 ga ko'ra u ruxsat etilgan va (13) o'rinli. (13) ga uchburchak tengsizligini va Lemma 3 dagi $|y(t) - y_0| \leq \sigma \Phi_\alpha(t)$ bahoni qo'llaymiz:

$$|y(t) - x(t)| \leq d + \left(1 - \frac{\rho}{\sigma}\right) |y(t) - y_0| \\ \leq d + \left(1 - \frac{\rho}{\sigma}\right) \sigma \Phi_\alpha(t) = D(t).$$

Bu baho qochuvchining istalgan ruxsat etilgan boshqaruvi uchun o'rinli, demak (16) dagi o'ng tomon $D(t)$ dan katta emas.

Yakun. Har doim $\sup \inf \leq \inf \sup$ bo'lgani uchun ikkala tomon $D(t)$ ga teng. Baholar aynan (10) va (12) strategiyalarda erishilgani ularning optimalligini bildiradi. $\square$

Izoh 1. Teorema 2 ning mazmuni: qochuvchi $D(t)$ masofani kafolatlay oladi, lekin bundan ko'prog'ini kafolatlay olmaydi; quvuvchi masofani $D(t)$ dan oshirmaslikni kafolatlay oladi, lekin kamroqni kafolatlay olmaydi. Demak, $D(t)$ — o'yinning har bir paytdagi qiymati. Bunda ikkala strategiya ham juda sodda: qochuvchi quvuvchidan to'g'ri qochadi, quvuvchi esa qochuvchining har bir harakatini ρ/σ nisbatda takrorlaydi. $\sigma = \rho$ bo'lganda $D(t) \equiv d$ va mutanosib quvish "ko'zgu" strategiyasiga ($u = v$) aylanadi: quvuvchi masofani doimiy d da ushlab turadi, qochuvchi esa uni kamaytirishga yo'l qo'ymaydi.

Natija 1 (universallik). Qochish sharti $\sigma \geq \rho$ kasr tartib α ga ham, Gronuoll parametri k ga ham bog'liq emas. Optimal strategiyalar (10) va (12) ning ko'rinishi ham α ga bog'liq emas. α va k faqat kafolatlangan masofa $D(t)$ ning vaqt bo'yicha o'sish qonuniga ta'sir qiladi.

Isbot. Teorema 1 va 2 ning isbotlarida α va k faqat $\Phi_\alpha(t)$ orqali qatnashadi, xulosalar esa Φ_α ning faqat musbatligi va o'suvchanligidan (Lemma 2(b)) foydalanadi. $\square$

Natija 2 (o'yinning to'liq manzarasi). Gronuoll tipidagi cheklovli kasr tartibli oddiy harakat o'yinida qochish masalasi $\sigma \geq \rho$ da va faqat shundagina yechiladi. $\rho > \sigma$ bo'lganda esa quvuvchi Π -strategiya yordamida $(\rho - \sigma)\Phi_\alpha(t) = d$ tenglamaning ildizi T_α paytdan kechiktirmay tutishni ta'minlaydi [17].

Isbot. $\sigma \geq \rho$ holi — Teorema 1; $\rho > \sigma$ holi — [17] dagi Teorema 1. $\square$

Izoh 2 (xususiy hollar). Lemma 2(d) ga ko'ra $D(t)$ quyidagi ko'rinishlarni oladi:

— $\alpha = 1, k = 0$: $D(t) = d + (\sigma - \rho)t$ — klassik oddiy harakat o'yinining ma'lum natijasi;

— $\alpha = 1, k > 0$: $D(t) = d + (2(\sigma - \rho))/k(e^{kt/2} - 1)$ — Gronuoll cheklovli klassik o'yin [5];

— $k = 0, 0 < \alpha < 1$: $D(t) = d + (\sigma - \rho)t^\alpha/\Gamma(\alpha + 1)$ — geometrik cheklovli kasr o'yin; masofa t^α tartibda, ya'ni klassik holdan sekinroq o'sadi.

Umumiy holda (8) ga ko'ra

$$D(t) \geq d + (\sigma - \rho) \frac{t^\alpha}{\Gamma(\alpha + 1)}, \\ D(t) \leq d + (\sigma - \rho) \frac{t^\alpha}{\Gamma(\alpha + 1)} e^{kt/2}.$$

3.2. Muhokama va sonli misol

Olingan natijalarni umumlashtirib aytish mumkinki, Gronuoll tipidagi cheklov va kasr tartibli dinamika bitta quvuvchi va bitta qochuvchi o'yinining sifat manzarasini o'zgartirmaydi: $\rho > \sigma$ da tutish, $\sigma \geq \rho$ da qochish; optimal strategiyalar ham klassik holdagidek — Π -strategiya, uzoqlashish va mutanosib quvish. O'zgaradigan narsa — miqdoriy ko'rsatkichlar: tutish vaqti va kafolatlangan masofa samarali vaqt Φ_α orqali ifodalanadi. Bunda kasr tartib α ning kamayishi masofaning o'sishini sekinlashtiradi, Gronuoll parametri k ning o'sishi esa tezlashtiradi.

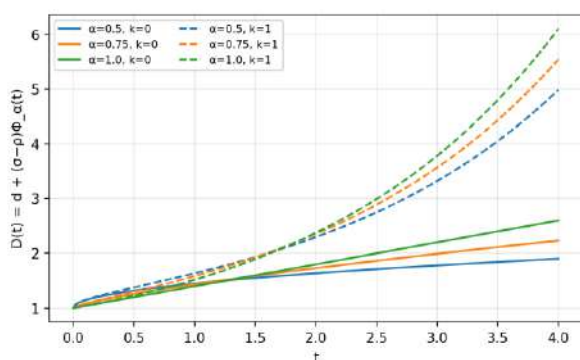
Misol 1. $d = 1, \sigma = 1, \rho = 0,6$ bo'lsin. U holda $D(t) = 1 + 0,4 \Phi_\alpha(t)$. Uning $\alpha \in \{0,5; 0,75; 1\}$ va $k \in \{0; 1\}$ uchun grafiklari 1-rasmda keltirilgan. Masalan, $t = 2$ paytda:

— $\alpha = 1, k = 0$: $D = 1 + 0,4 \cdot 2 = 1,80$;

— $\alpha = 0,5, k = 0$: $D = 1 + 0,4 \cdot 2^{0,5}/\Gamma(1,5) \approx 1,64$;

— $\alpha = 1, k = 1$: $D = 1 + 0,8(e - 1) \approx 2,37$.

Qochish fakti barcha hollarda saqlanadi, kafolatlangan masofa esa sezilarli farq qiladi.



1-rasm. Kafolatlangan masofa $D(t) = d + (\sigma - \rho)\Phi_\alpha(t)$;
 $d = 1$, $\sigma = 1$, $\rho = 0,6$.

IV. Xulosa.

(Natija 2). Keyingi tadqiqotlar uchun tabiiy yoʻnalishlar — koʻp quvvuchidan qochish va kasr dinamikada “hayot chizigʻi” oʻyini.

Adabiyotlar

- [1] Isaacs R. *Differential Games*, New York: John Wiley & Sons, 1965, 384 p.
- [2] Pontryagin L.S. *Izbrannye nauchnye trudy*, vol. 2, Moscow: Nauka, 1988, 576 p.
- [3] Kumkov S.S., Le Ménec S., Patsko V.S. Zero-sum pursuit-evasion differential games with many objects: survey of publications, *Dynamic Games and Applications*, 2017, vol. 7, issue 4, pp. 609–633. <https://doi.org/10.1007/s13235-016-0209-z>
- [4] Azamov A.A., Samatov B.T. *Π-strategy. An elementary introduction to the theory of differential games*, Tashkent: National University of Uzbekistan, 2000, 32 p.
- [5] Samatov B.T., Ibragimov G.I., Khodjibayeva I.V. Pursuit-evasion differential games with Grönwall type constraints on controls, *Ural Mathematical Journal*, 2020, vol. 6, issue 2, pp. 95–107. <https://doi.org/10.15826/umj.2020.2.010>
- [6] Samatov B.T., Akbarov A.Kh., Zhuraev B.I. Pursuit-evasion differential games with Grönwall constraints on controls, *Izvestiya Instituta Matematiki i Informatiki Udmurtskogo Gosudarstvennogo Universiteta*, 2022, vol. 59, pp. 67–84. <https://doi.org/10.35634/2226-3594-2022-59-06>
- [7] Ibragimov G.I., Yusupov I.M., Ferrara M. Evasion differential game of multiple coordinated evaders and multiple slow pursuers, *Chelyabinsk Physical and Mathematical Journal*. 2026. Vol. 11, iss. 2. P. 302–314. <https://doi.org/10.3390/math11020374>
- [7] Ibragimov G.I., Yusupov I.M., Ferrara M. Optimal pursuit game of two pursuers and one evader with the Grönwall-type constraints on controls, *Mathematics*, 2023, vol. 11, issue 2, Article 374. DOI: 10.47475/2500-0101-2026-11-2-302-314

Maqolada Kaputo kasr tartibli dinamikali va Gronuoll tipidagi cheklovli oddiy harakat differensial oʻyinida bitta quvvuch va bitta qochuvchi uchun qochish masalasi toʻliq yechildi. Qochuvchining imkoniyati quvvuchinikidan kam boʻlmaganda uzoqlashish strategiyasi qochishni kafolatlaydi va masofani $D(t) = d + (\sigma - \rho)\Phi_\alpha(t)$ dan kam boʻlmasligini taʼminlaydi (Teorema 1). Quvvuchining mutanosib quvish strategiyasi masofani shu miqdordan oshirmaydi, shuning uchun $D(t)$ — kafolatlangan masofa oʻyining qiymati, ikkala strategiya esa optimal (Teorema 2). Qochish sharti kasr tartibga va Gronuoll parametriga bogʻliq emas (Natija 1); [17] bilan birgalikda oʻyinning toʻliq manzarasi olindi: qochish $\sigma \geq \rho$ da va faqat shundagina mumkin

- [8] Ibragimov G., Egamberganova O., Alias I.A., Luckraz S. On some new results in a pursuit differential game with many pursuers and one evader, *AIMS Mathematics*, 2023, vol. 8, issue 3, pp. 6581–6589. <https://doi.org/10.3934/math.2023332>
- [9] Rilwan J., Ferrara M., Badakaya A.J., Pansera B.A. On pursuit and evasion game problems with Grönwall-type constraints, *Quality & Quantity*, 2023, vol. 57, pp. 5551–5562. <https://doi.org/10.1007/s11135-023-01616-9>
- [10] Samatov B.T., Akbarov A.Kh. Differential game with a “life line” under the Grönwall constraint on controls, *Vestnik of Saint Petersburg University. Applied Mathematics. Computer Science. Control Processes*, 2024, vol. 20, issue 2, pp. 265–280. <https://doi.org/10.21638/spbu10.2024.211>
- [11] Grönwall T.H. Note on the derivatives with respect to a parameter of the solutions of a system of differential equations, *Annals of Mathematics*, 1919, vol. 20, issue 4, pp. 292–296. <https://doi.org/10.2307/1967124>
- [12] Kilbas A.A., Srivastava H.M., Trujillo J.J. *Theory and Applications of Fractional Differential Equations*, Amsterdam: Elsevier, 2006, 523 p.
- [13] Podlubny I. *Fractional Differential Equations*, San Diego: Academic Press, 1999, 340 p.
- [14] Gorenflo R., Kilbas A.A., Mainardi F., Rogosin S.V. *Mittag-Leffler Functions, Related Topics and Applications*, Berlin: Springer, 2014, 443 p. <https://doi.org/10.1007/978-3-662-43930-2>
- [15] Chikrii A.A., Matychyn I.I. Game problems for fractional-order systems, in: *New Trends in Nanotechnology and Fractional Calculus Applications*, Dordrecht: Springer, 2010, pp. 233–241. https://doi.org/10.1007/978-90-481-3293-5_20

On automorphisms of two-dimensional permuted lotka-volterra algebras

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Abstract: We study the automorphisms of genetic algebras associated with permuted Lotka-Volterra operators on low-dimensional simplex. We obtain complete descriptions of the automorphism of such algebras of dimensions two.

Аннотация: Исследуются автоморфизмы генетических алгебр, связанных с переставленными операторами Лотки — Вольтерры на симплексе низкой размерности. Получено полное описание автоморфизмов таких алгебр размерности два.

Annotatsiya: Biz past o'lchamli simpleksda qayta tartiblangan Lotka-Volterra operatorlari bilan bog'liq genetik algebralarning avtomorfizmlarini o'rganamiz. Bunday ikki o'lchamli algebralarning avtomorfizmlarining to'liq tavsifi olingan.

Keywords: genetic algebra, quadratic stochastic operator, non-Volterra operator, quadratic homeomorphisms, automorphisms

I. Introduction

It is known that a biological population defined as a community of organisms closed with respect to reproduction. A state of a population is a probability distribution of the different organisms types. One of the central problems in mathematical biology is to study the evolution of a given population. The mathematical methods applied to study this problem involve dynamical systems, probability theory, stochastic processes and (non)-associative algebras.

Let us consider a biological population. Assume that the population consists from m species (or traits) which are denote by $E = \{1, 2, \dots, m\}$ and each individual in this population belongs to precisely one species (genotype). Let $\mathbf{x}^{(0)} = (x_1^{(0)}, \dots, x_m^{(0)})$ be a probability distribution of species at an initial state of the population. Denote by $p_{ij,k}$ a probability that individuals in the i^{th} and j^{th} species interbreed to produce an individual from k^{th} species. Then a probability distribution $\mathbf{x}^{(1)} = (x_1^{(1)}, \dots, x_m^{(1)})$ of the species in the first generation can be found by the rule of a total probability, i.e.,

$$x_k^{(1)} = \sum_{i,j=1}^m p_{ij,k} x_i^{(0)} x_j^{(0)}, \quad k \in E.$$

The last defines a correspondence $\mathbf{x}^{(0)} \rightarrow \mathbf{x}^{(1)}$ which is called *the evolution operator* or *quadratic stochastic operator* (QSO). In other words, a QSO describes a distribution of the next generation when we know the distribution of the current generation was given. The applications of QSO to population genetics were given in [1,6,7,12]. In [5] the recent results and open problems in the theory of QSOs were given.

On the other hand, each QSO defines an algebraic structure with a genetic significance on the vector space $\mathbb{R}^m$ containing the simplex (see next section for definitions). This algebraic structure is called *genetic algebra*. They are generally commutative but non-associative, yet they are not necessarily Lie, Jordan, or alternative algebras. Additionally, many of the algebraic properties of these structures have genetic significance. For example, a more modern use of the genetic algebra theory to self fertilization can be found in [7,8]. So, it is the interplay between the purely mathematical structure and the

corresponding genetic properties that makes this subject so fascinating. We refer to [22] for the comprehensive reference.

The general theory of groups contains a developed field of the groups of automorphism of algebraic structures. The research of the automorphism groups of any algebraic object is one of natural and important problems of this theory. Note that the automorphism groups of the algebras associated with the Lotka-Volterra operators (shortly Lotka-Volterra algebras) are studied (see [2,23]). Therefore, it is quite natural to originate these investigations for algebras associated with the other quadratic operators whose structure is fairly well studied [12,15,16]. The paper is organized as follows. In Section 2 we recall necessary definitions from the theory of genetic algebras. In Section 3 we describe the group of automorphisms of the 2D genetic algebras associated with PLV operators. In Section 4 we give the proof of main results given previous sections.

Preliminaries

Let us recall the definition of an evolution algebra of a free population following [11].

Let $E = \{1, \dots, m\}$. By $\{\mathbf{e}_i\}_{i \in E}$ we denote the standard basis in $\mathbb{R}^m$, i.e., $\mathbf{e}_i = (\delta_{i1}, \dots, \delta_{im})$, where δ_{ij} is the Kronecker's delta. Consider the simplex:

$$S^{m-1} = \left\{ \mathbf{x} \in \mathbb{R}^m : \mathbf{x} = (x_1, x_2, \dots, x_m), x_i \geq 0, \forall i \in E, \sum_{i \in E} x_i = 1 \right\}.$$

A quadratic stochastic operator (QSO) is a mapping of S^{m-1} into itself of the form

$$V: x'_k = \sum_{i,j \in E} p_{ij,k} x_i x_j, \quad k \in E, \quad (1)$$

where $p_{ij,k}$ are inheritance coefficients, which satisfy the following conditions:

$$p_{ij,k} \geq 0, \quad p_{ij,k} = p_{ji,k}, \quad \sum_{k \in E} p_{ij,k} = 1, \quad i, j, k \in E. \quad (2)$$

Let V be a QSO and for arbitrary $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, we define a multiplication rule on $\mathbb{R}^m$ by

$$(\mathbf{x} \circ_V \mathbf{y})_k = \sum_{i,j \in E} p_{ij,k} x_i y_j, \quad k \in E, \quad (3)$$

where $\mathbf{x} = (x_1, \dots, x_m), \mathbf{y} = (y_1, \dots, y_m) \in \mathbb{R}^m$ (see [7]).

Using (2) it is easy to see that the multiplication (3) is commutative. Algebraic properties of such algebras were studied in [7,9,10,12-19,21,22]. In general, the genetic algebra is not necessarily to be associative. The multiplication (3) in the canonical basis can be represented as follows $\mathbf{e}_i \circ_V \mathbf{e}_j = \sum_{k \in E} p_{ij,k} \mathbf{e}_k$, $i, j \in E$. Thus, we identify the coefficients (2) as the structure of an algebra, i.e., a bilinear mapping of $\mathbb{R}^m \times \mathbb{R}^m$ to $\mathbb{R}^m$. It turns out that the multiplication can be given terms of QSO (1) as

$$\mathbf{x} \circ_V \mathbf{y} = \sum_{i,j,k \in E} (p_{ij,k} x_i y_j) \mathbf{e}_k = \frac{1}{4} (V(\mathbf{x} + \mathbf{y}) - V(\mathbf{x} - \mathbf{y})), \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^m.$$

One can check that $\mathbf{x} \circ_V \mathbf{x} = \mathbf{x}^2 = V(\mathbf{x})$ for any $\mathbf{x} \in S^{m-1}$. This algebraic interpretation is useful, e.g., a state $\mathbf{x}$ is an equilibrium precisely when $\mathbf{x}$ is an idempotent element of the standard simplex S^{m-1} . We write $\mathbf{x}^{[n]}$ for the power $(\dots (\mathbf{x}^2)^2 \dots)$ (n times) with $\mathbf{x}^{[0]} \equiv \mathbf{x}$, then the orbit for initial $\mathbf{x}$ is $V^n(\mathbf{x}) = \mathbf{x}^{[n]}$.

The algebra $\mathcal{A}_V$ (i.e., the pair $(\mathbb{R}^m, \circ_V)$) associated by the operator (1) is called the *genetic algebra*. Notice that algebras $\mathcal{A}_V$ associated by Lotka-Volterra operators are called the *Lotka-Volterra algebras* and their properties have been studied in [2,4,20].

Let π be a permutation of the set E . Define a map $T_\pi: S^{m-1} \rightarrow S^{m-1}$ by $T_\pi(\mathbf{x}) = (x_{\pi(1)}, \dots, x_{\pi(m)})$. Note that any stochastic linear bijection of the simplex coincides with T_π for some π . Two QSOs V and W are called *topologically conjugate* if there is a permutation π such that $T_\pi^{-1}VT_\pi = W$, the last one is denoted by $V \sim W$.

Let V, W be QSOs on S^{m-1} and $\mathcal{A}_V, \mathcal{A}_W$ be the associated genetic algebras. These algebras are called *stochastically isomorphic* if there is a stochastic linear bijection $\psi: \mathcal{A}_V \rightarrow \mathcal{A}_W$ such that $\psi(\mathbf{x} \circ_V \mathbf{y}) = \psi(\mathbf{x}) \circ_W \psi(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{A}_V$.

The next theorem about connection of between the conjugation of QSOs V, W and associated genetic algebras $\mathcal{A}_V, \mathcal{A}_W$.

Theorem 1 [15]. *Let V and W be QSOs. Then the following assertions are equivalent:*

- *V and W are conjugate;*
- *the genetic algebras $\mathcal{A}_V$ and $\mathcal{A}_W$ are stochastically isomorphic.*

It is known that automorphisms of algebras are an important tool for understanding algebraic structures, with applications spanning theoretical and practical domains including physics, computer science, and pure mathematics. Let $\mathcal{A}_V$ be a genetic algebra. Recall that a linear transformation A of the algebra $\mathcal{A}_V$ is an *endomorphism* if $A(\mathbf{x} \circ_V \mathbf{y}) = A(\mathbf{x}) \circ_V A(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{A}_V$. We say that a bijective endomorphism of the algebra $\mathcal{A}_V$ is an *automorphism* of the algebra $\mathcal{A}_V$.

We use the following steps to obtain all automorphisms of the given algebra $\mathcal{A}_V$:

Step 1. Using (3) from the multiplications $\mathbf{e}_i \circ_V \mathbf{e}_j$ for all $i, j \in E$, we construct a system consisting from the equations $A(\mathbf{e}_i \circ_V \mathbf{e}_j) = A(\mathbf{e}_i) \circ_V A(\mathbf{e}_j)$ for all $i, j \in E$;

Step 2. We solve the subsystem consisting from the equations $A(\mathbf{e}_i \circ_V \mathbf{e}_i) = A(\mathbf{e}_i) \circ_V A(\mathbf{e}_i)$ for all $i \in E$ and from the obtained solutions (matrices) A we select invertible ones, that is those solutions for which $\det A \neq 0$;

Step 3. From the invertible solutions A we choose the solutions which satisfy the subsystem of the equations $A(\mathbf{e}_i \circ_V \mathbf{e}_j) = A(\mathbf{e}_i) \circ_V A(\mathbf{e}_j)$ for all $i \neq j \in E$;

Step 4. The set of all solutions selected in the step 3 is the set of automorphisms of $\mathcal{A}_V$.

Automorphisms of two-dimensional genetic algebras associated by permuted Lotka-Volterra operators

Let us consider the following Lotka-Volterra QSO defined on the S^1

$$V: \begin{cases} x'_1 = x_1^2 + (1-a)x_1x_2, \\ x'_2 = x_2^2 + (1+a)x_1x_2, \end{cases} \quad -1 \leq a \leq 1.$$

It is known that the set $E = \{1, 2\}$ has the permutations $\pi_1 = e$ and $\pi_2 = (21)$, where e is the identity permutation. It is easy to see that for the permutation π_1 one has $T_{\pi_1} = Id$ and for π_2 we have $T_{\pi_2} = Id$, where Id is the identity map. In [15] showed that there are two quadratic homeomorphisms of S^1 and they defined by $V_{2,1} = T_{\pi_1}V$ and $V_{2,2} = T_{\pi_2}V$. Therefore it follows that

$$V_{2,1}: \begin{cases} x'_1 = x_1^2 + (1-a)x_1x_2, \\ x'_2 = x_2^2 + (1+a)x_1x_2, \end{cases} \quad V_{2,2}: \begin{cases} x'_1 = x_2^2 + (1+a)x_1x_2, \\ x'_2 = x_1^2 + (1-a)x_1x_2, \end{cases} \quad -1 \leq a \leq 1. \quad (4)$$

Note that $V_{2,1}$ is a Lotka-Volterra operator and $V_{2,2}$ is a permuted Lotka-Volterra operator. We shall investigate the genetic algebras $\mathcal{A}_{V_{2,1}}$ and $\mathcal{A}_{V_{2,2}}$ associated by quadratic homeomorphisms $V_{2,1}$ and $V_{2,2}$, respectively.

3.1. Automorphisms of the 2D algebra $\mathcal{A}_{V_{2,1}}$. Consider the operator $V_{2,1}$ (4). Then the algebra $\mathcal{A}_{V_{2,1}} := \mathcal{A}_{V_{2,1}}(a)$, $a \in [-1, 1]$ has the multiplication table:

$$\mathbf{e}_1 \circ_{V_{2,1}} \mathbf{e}_1 = \mathbf{e}_1, \quad \mathbf{e}_2 \circ_{V_{2,1}} \mathbf{e}_2 = \mathbf{e}_2, \quad 2\mathbf{e}_1 \circ_{V_{2,1}} \mathbf{e}_2 = 2\mathbf{e}_2 \circ_{V_{2,1}} \mathbf{e}_1 = (1-a)\mathbf{e}_1 + (1+a)\mathbf{e}_2.$$

Theorem 2. For the 2D algebra $\mathcal{A}_{V_{2,1}}(a)$ over the field $\mathbb{R}$ the following statements hold:

- i) $\text{Aut}(\mathcal{A}_{V_{2,1}}(0)) = \{A_{\alpha, \beta} \cong \text{Aff}(1, \mathbb{R}), \text{ where } A_{\alpha, \beta}(\mathbf{e}_1) = \alpha\mathbf{e}_1 + (1-\alpha)\mathbf{e}_2, A_{\alpha, \beta}(\mathbf{e}_2) = \beta\mathbf{e}_1 + (1-\beta)\mathbf{e}_2, \alpha \neq \beta \in \mathbb{R};$
- ii) $\text{Aut}(\mathcal{A}_{V_{2,1}}(1)) = \{Id, \sigma\} \cong \mathbb{Z}_2, \text{ where } \sigma(\mathbf{e}_1) = \mathbf{e}_1, \sigma(\mathbf{e}_2) = \mathbf{e}_1 - \mathbf{e}_2;$
- iii) $\text{Aut}(\mathcal{A}_{V_{2,1}}(-1)) = \{Id, \tau\} \cong \mathbb{Z}_2, \text{ where } \tau(\mathbf{e}_1) = -\mathbf{e}_1 + \mathbf{e}_2, \tau(\mathbf{e}_2) = \mathbf{e}_2;$
- iv) if $a \in [-1, 1] \setminus \{-1, 0, 1\}$ then $\text{Aut}(\mathcal{A}_{V_{2,1}}(a)) = \{Id\}.$

3.2. Automorphisms of the 2D algebra $\mathcal{A}_{V_{2,2}}$. We consider the operator $V_{2,2}$ defined by (4). Then the algebra $\mathcal{A}_{V_{2,2}} := \mathcal{A}_{V_{2,2}}(a)$ has the multiplication table:

$$\mathbf{e}_1 \circ_{V_{2,2}} \mathbf{e}_1 = \mathbf{e}_2, \quad \mathbf{e}_2 \circ_{V_{2,2}} \mathbf{e}_2 = \mathbf{e}_1, \quad 2\mathbf{e}_1 \circ_{V_{2,2}} \mathbf{e}_2 = 2\mathbf{e}_2 \circ_{V_{2,2}} \mathbf{e}_1 = (1+a)\mathbf{e}_1 + (1-a)\mathbf{e}_2.$$

Theorem 3. For the 2D algebra $\mathcal{A}_{V_{2,2}}(a)$ over the field $\mathbb{R}$ the following statements hold:

- i) $\text{Aut}(\mathcal{A}_{V_{2,2}}(0)) = \{A_\alpha \cong (\mathbb{R}^\times, \cdot), \text{ where } A_\alpha(\mathbf{e}_1) = \alpha\mathbf{e}_1 + (1-\alpha)\mathbf{e}_2, A_\alpha(\mathbf{e}_2) = (1-\alpha)\mathbf{e}_1 + \alpha\mathbf{e}_2, \alpha \in \mathbb{R} \setminus \{\frac{1}{2}\};$
- ii) if $a \neq 0$ then $\text{Aut}(\mathcal{A}_{V_{2,2}}(a)) = \{Id\}.$

Appendix

This section deals to proofs of Theorems 2, 3 given in the Sections 3. Denote by $\mathbb{I}_n$ the identity matrix of the dimension n .

Proof of Theorem 2. Let A be a linear automorphism of the algebra $\mathcal{A}_{V_{2,1}}$ and it follows $A(\mathbf{e}_i) = a_{i1}\mathbf{e}_1 + a_{i2}\mathbf{e}_2$, $i = 1, 2$. Then from the equations $A(\mathbf{e}_i \circ_{V_{2,1}} \mathbf{e}_j) = A(\mathbf{e}_i) \circ_{V_{2,1}} A(\mathbf{e}_j)$, $i, j = 1, 2$ we obtain the following system

$$\begin{cases} a_{11}^2 + (1-a)a_{11}a_{12} = a_{11}, & a_{12}^2 + (1+a)a_{11}a_{12} = a_{12}, \\ a_{21}^2 + (1-a)a_{21}a_{22} = a_{21}, & a_{22}^2 + (1+a)a_{21}a_{22} = a_{22}, \\ (1-a)a_{11} + (1+a)a_{21} = 2a_{11}a_{21} + (a_{11}a_{22} + a_{12}a_{21})(1-a), \\ (1-a)a_{12} + (1+a)a_{22} = 2a_{12}a_{22} + (a_{11}a_{22} + a_{12}a_{21})(1+a), \end{cases} \quad (6)$$

where $a \in [-1, 1]$.

We solve the system (6) following to steps mentioned in the Section 2.

From the 1st, 2nd (resp. 3rd, 4th) equations of (6) we have $(a_{k1} + a_{k2})^2 = a_{k1} + a_{k2}$, $k = 1, 2$ and so it follows either $a_{k1} + a_{k2} = 0$ or $a_{k1} + a_{k2} = 1$, $k = 1, 2$.

i) Let $a = 0$, i.e. we consider $\mathcal{A}_{V_{2,1}}(0)$. Then 1st, 2nd equations of (6) has form

$$a_{11}(1 - (a_{11} + a_{12})) = 0, \quad a_{12}(1 - (a_{11} + a_{12})) = 0. \quad (7)$$

It is easy to verify that if $a_{11} + a_{12} = 1$ (resp. $a_{11} + a_{12} = 0$) then the system (7) has the unique solution $a_{11} = \alpha$, $a_{12} = 1 - \alpha$, $\alpha \in \mathbb{R}$ (resp. $a_{11} = a_{12} = 0$). Similarly from the 3rd and 4th equations of (6) we obtain that if $a_{21} + a_{22} = 1$ (resp. $a_{21} + a_{22} = 0$) then we get $a_{21} = \beta$, $a_{22} = 1 - \beta$, $\beta \in \mathbb{R}$ (resp. $a_{21} = a_{22} = 0$). Therefore, we have

$$A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ \beta & 1 - \beta \end{pmatrix}, \quad A_3 = \begin{pmatrix} \alpha & 1 - \alpha \\ 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} \alpha & 1 - \alpha \\ \beta & 1 - \beta \end{pmatrix}.$$

Evidently, that $\det(A_i) = 0, i = 1, 2, 3$ and $\det(A_4) = \alpha - \beta$. Besides one can check the entries of A_4 satisfy the rest the last two equations of (6). Here, $\alpha \neq \beta$, because if $\alpha = \beta$ then $\det(A_4) = 0$.

ii) Let $a \neq 0$, i.e. we consider $\mathcal{A}_{V_{2,1}}(a)$. Then 1st, 2nd equations of (6) imply

$$a_{11}(1 - (a_{11} + a_{12}) + aa_{12}) = 0, \quad a_{12}(1 - (a_{11} + a_{12}) - aa_{11}) = 0. \quad (8)$$

If $a_{11} + a_{12} = 0$ then from (8) we have either $a_{11} = a_{12} = 0$ or $a_{11} = -a_{12} = \frac{1}{a}$. Also from (8) we get either $a_{11} = 1, a_{12} = 0$ or $a_{11} = 0, a_{12} = 1$ when $a_{11} + a_{12} = 1$. Similarly the 3rd and 4th equations of (6) has form $a_{21}(1 - (a_{21} + a_{22}) + aa_{22}) = 0, a_{22}(1 - (a_{21} + a_{22}) - aa_{21}) = 0$ and the latter has the solutions $a_{21} = a_{22} = 0, a_{21} = -a_{22} = \frac{1}{a}, a_{21} = 1, a_{22} = 0, a_{21} = 0, a_{22} = 1$.

Using these solutions we get $A_5 = \begin{pmatrix} \frac{1}{a} & -\frac{1}{a} \\ 1 & 0 \end{pmatrix}, A_6 = \begin{pmatrix} \frac{1}{a} & -\frac{1}{a} \\ 0 & 1 \end{pmatrix}, A_7 = \begin{pmatrix} 1 & 0 \\ \frac{1}{a} & -\frac{1}{a} \end{pmatrix}, A_8 = \begin{pmatrix} 0 & 1 \\ \frac{1}{a} & -\frac{1}{a} \end{pmatrix},$

$A_9 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_{10} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with $\det(A_i) \neq 0, i = 6, 7, 8, 9, 10$. One has that A_5, A_8, A_{10} do not satisfy (6), A_6 (resp. A_7) satisfies (6) when $a = -1$ (resp. $a = 1$), $A_9 = \mathbb{I}_2$.

Proof of Theorem 3. Let A be a linear automorphism of $\mathcal{A}_{V_{2,2}}$ and $A(\mathbf{e}_i) = a_{i1}\mathbf{e}_1 + a_{i2}\mathbf{e}_2, i = 1, 2$. Then the equations $A(\mathbf{e}_i \circ_{V_{2,2}} \mathbf{e}_j) = A(\mathbf{e}_i) \circ_{V_{2,2}} A(\mathbf{e}_j), i, j = 1, 2$ imply the following system

$$\begin{cases} a_{12}^2 + (1 + a)a_{11}a_{12} = a_{21}, & a_{11}^2 + (1 - a)a_{11}a_{12} = a_{22}, \\ a_{22}^2 + (1 + a)a_{21}a_{22} = a_{11}, & a_{21}^2 + (1 - a)a_{21}a_{22} = a_{12}, \\ (1 + a)a_{11} + (1 - a)a_{21} = 2a_{12}a_{22} + (a_{11}a_{22} + a_{12}a_{21})(1 + a), \\ (1 + a)a_{12} + (1 - a)a_{22} = 2a_{11}a_{21} + (a_{11}a_{22} + a_{12}a_{21})(1 - a). \end{cases} \quad (9)$$

where $a \in [-1, 1]$.

We solve the system (9) following to steps mentioned in the Section 2.

From the 1st four equations of (9) we have $(a_{11} + a_{12})^4 = (a_{21} + a_{22})^2 = a_{11} + a_{12}$ and it follows either $a_{11} + a_{12} = 0$ or $a_{11} + a_{12} = 1$ (resp. $a_{21} + a_{22} = 0$ or $a_{21} + a_{22} = 1$). Also we get that the values $a_{11} + a_{12}$ and $a_{21} + a_{22}$ are both either 0 or 1, simultaneously.

i) Let $a = 0$, that is we consider the algebra $\mathcal{A}_{V_{2,2}}(0)$. Then it is easy to check that if $a_{11} + a_{12} = 0$ then (9) has the unique solution $a_{ij} = 0, i, j = 1, 2$. Also one has if $a_{11} + a_{12} = 1$ then (9) has a solution $a_{11} = a_{22} = \alpha, a_{12} = a_{21} = 1 - \alpha, \alpha \in \mathbb{R}$. Hence we have if $\alpha \in \mathbb{R} \setminus \{\frac{1}{2}\}$ then $A_{11} = \begin{pmatrix} \alpha & 1 - \alpha \\ 1 - \alpha & \alpha \end{pmatrix}$ with $\det(A_{11}) \neq 0$ and the map $A(\mathbf{e}_1) = \alpha\mathbf{e}_1 + (1 - \alpha)\mathbf{e}_2, A(\mathbf{e}_2) = (1 - \alpha)\mathbf{e}_1 + \alpha\mathbf{e}_2$ is an automorphism.

ii) Let $a \neq 0$ in $\mathcal{A}_{V_{2,2}}(a)$. Then the 1st four equations of (9) imply either $a_{ij} = 0, i, j = 1, 2$ or $a_{i1} = -a_{i2} = -\frac{1}{a}, i = 1, 2$ when $a_{11} + a_{12} = 0$ and therefore in this case obtained matrices have zero determinants.

If $a_{11} + a_{12} = 1$ then from the 1st four equations of (9) we have A_9, A_{10} ,

$$A_{12} = \begin{pmatrix} \frac{a - 2 - q(a)}{2a} & \frac{a + 2 + q(a)}{2a} \\ \frac{(a + 2 + q(a))(a - q(a))}{4a} & \frac{(2 - a + q(a))(a + q(a))}{4a} \end{pmatrix},$$

$$A_{13} = \begin{pmatrix} \frac{a-2+q(a)}{2a} & \frac{a+2-q(a)}{2a} \\ \frac{(a+2-q(a))(a+q(a))}{4a} & \frac{(a-2+q(a))(-a+q(a))}{4a} \end{pmatrix}, \quad q(a) = \sqrt{a^2 + 4}.$$

Evidently that $\det A_{12} = \det A_{13} = 0$. Also it holds $\det A_{10} \neq 0$, but it does not satisfy 5th and 6th equations of (9).

References

- [1]. Bernstein, S.: Solution of a mathematical problem connected with the theory of heredity. *Ann. Math. Statistics.* 13(1), 53–61 (1942)
- [2]. Fernandez, J.C.G., Garcia, C.I.: On Lotka-Volterra algebras. *Jour. Alg. Appl.* 18(10), 245–258 (2019)
- [3]. Ganikhodzhaev, R.N., Eshmatova, D.B.: Quadratic automorphisms of a simplex and the asymptotic behavior of their trajectories. *Vladikavkaz. Mat. Zh.* 8, 12–28 (2006) (in Russian)
- [4]. Ganikhodzhaev, R.N., Mukhamedov, F., Pirnapasov, A., Qaralleh, I.: Genetic Volterra algebras and their derivations. *Comm. Algebra.* 46(2), 1353–1366 (2018)
- [5]. Ganikhodzhaev, R.N., Mukhamedov, F., Rozikov, U.: Quadratic stochastic operators and processes: results and open problems. *Infin. Dim. Anal. Quan. Probab. Relat. Top.* 14, 279–335 (2011)
- [6]. Harrison, D.K. Commutative non-associative algebras and cubic forms. *J. Algebra.* 32 (3), 518–528 (1974)
- [7]. Holgate, P.: Genetic algebras satisfying Bernstein's stationarity principle. *J. London Math. Soc.* 9, 613–623 (1975)
- [8]. Holgate, P.: Selfing in genetic algebras. *J. Math. Biol.* 6, 197–206 (1978)
- [9]. Jamilov, U.U., Ladra, M.: F-evolution algebra. *Filomat.* 30, 2637–2652 (2016)
- [10]. Jamilov, U.U., Ladra, M.: Evolution algebras and dynamical systems of a worm propagation model. *Lin. Multilin. Alg.* 70(19), 4097–4116 (2022)
- [11]. Lotka, A.J.: Undamped oscillations derived from the law of mass action. *J. Amer. Chem. Soc.* 42, 1595–1599 (1920)
- [12]. Lyubich, Y.I.: Mathematical structures in population genetics, vol. 22 of *Biomathematics*, Springer-Verlag, Berlin, (1992)
- [13]. Mukhamedov, F., Embong, A.F.: Infinite dimensional orthogonality preserving nonlinear Markov operators. *Lin. Multilin. Alg.* 69(3), 526–550 (2021)
- [14]. Mukhamedov, F., Khakimov, O., Qaralleh, I.: Classification of nilpotent evolution algebras and extensions of its derivations, *Commun. Alg.* 48(10), 4155–4169 (2020)
- [15]. Mukhamedov, F., Jamilov, U., Ibragimov, M.: Genetic algebras associated with permuted Lotka-Volterra operators. *Commun. Alg.* 52(12), 5224–5240 (2024)
- [16]. Mukhamedov, F., Qarallah, I., Qaisar, T., Hasan, M.A.: Genetic algebras associated with $\xi^{(a)}$ -quadratic stochastic operators. *Entropy.* 25(6), a.n. 934 (2023)
- [17]. Mukhamedov, F., Qaralleh, I., Pirnapasov, A.: On genetic and evolution algebras in dimension three. *Inter. J. Alg.* 10(7), 327–334 (2016)
- [18]. Mukhamedov, F., Syam, S.M., Qaralleh, I.: A flow of quantum genetic Lotka-Volterra algebras on $M_2(C)$. *J. Geom. Phys.* 190, a.n. 104854 (2023)
- [19]. Mukhamedov, F., Taha, M.H.M.: On Volterra and orthogonality preserving quadratic stochastic operators. *Miskolc. Math. Notes.* 17(1), 457–470 (2016)
- [20]. Qaralleh, I., Mukhamedov, F.: Volterra evolution algebras and their graphs. *Lin. Multilin. Alg.* 69, 2228–2244 (2021)
- [21]. Rozikov, U.A.: Population dynamics: algebraic and probabilistic approach. *World Sci. Publ.*, Singapore, (2020)
- [22]. Worz-Busekros, A.: Algebras in Genetics. *Lect. Notes in Biomathematics*, Vol. 36. Springer-Verlag, Berlin (1980)
- [23]. Yoon, S.I.: Automorphisms of Lotka-Volterra algebras. *Comm. Korean Math. Soc.* 12, 45–50 (1997)

An explicit error estimate for the fifth-order Taylor method for the Cauchy problem for quadratic systems

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Аннотация. Выводится неравенство, оценивающее порядок точности метода приближённого решения задачи Коши для квадратичных динамических систем. Рассматриваемая схема построена по первым пяти членам ряда Тейлора точного решения. Используя конечность формулы Тейлора для квадратичного отображения, мы точно вычисляем невязку приближённого решения, доказываем неравенство для погрешности и получаем явные выражения для соответствующих коэффициентов.

Ключевые слова: Динамическая система, задача Коши, квадратичная система, приближённое решение, формула Тейлора, лемма Гронуолла, порядок точности, оценка погрешности.

Abstract. We derive an inequality that estimates the order of accuracy of an approximate solution method for the Cauchy problem for quadratic dynamical systems. The scheme under consideration is built from the first five terms of the Taylor series of the exact solution. Using the fact that the Taylor formula of a quadratic mapping is finite, we compute the residual of the approximate solution exactly, prove the error inequality and obtain explicit expressions for the corresponding coefficients.

Keywords: Dynamical system, Cauchy problem, quadratic system, approximate solution, Taylor formula, Gronwall lemma, order of accuracy, error estimation.

Annotatsiya. Kvadratlik dinamik sistemalar uchun Koshi masalasini taqribiy yechish usulining aniqlik tartibini baholovchi tengsizlik keltirib chiqariladi. Qaralayotgan sxema aniq yechimning Teylor qatorining dastlabki beshta hadi asosida qurilgan. Kvadratlik akslantirish uchun Teylor formulasi chekli ekanligidan foydalanib, taqribiy yechimning tafovuti aniq hisoblanadi, xatolik uchun tengsizlik isbotlanadi va mos koeffitsiyentlar uchun oshkor ifodalar olinadi.

Kalit so'zlar: Dinamik sistema, Koshi masalasi, kvadratlik sistema, taqribiy yechim, Teylor formulasi, Gronuoll lemmasi, aniqlik tartibi, xatolikni baholash.

1. INTRODUCTION

We consider the Cauchy problem for a system of ordinary differential equations

$$\dot{x}(t) = f[x(t)], \quad x(0) = x_0. \quad (1)$$

Problems of this type arise throughout mathematical modelling, and their approximate solution is a classical subject. Here $x \in D \subset \mathbb{R}^d$, D is a convex domain, and $f: D \rightarrow \mathbb{R}^d$ is a quadratic mapping. The quality of an approximate solution method is characterized first of all by its order of accuracy. If $x_*(t)$ is the exact solution of problem (1) and $\tilde{x}(t)$, $0 \leq t \leq T$, is an approximate one, the order of accuracy is expressed by an inequality

$$\sup_{0 \leq t \leq T} |x_*(t) - \tilde{x}(t)| \leq Ce^{LT} h^s, \quad (2)$$

where L is a Lipschitz constant, h is the step of the grid on which the approximate solution is computed, and s is the order of the method. This problem is treated in a large number of works; see, e.g., [5, 6, 7, 12].

One-step approximate solution schemes are usually divided into two main groups: schemes based on the Taylor formula and schemes based on Runge–Kutta methods. Both are widely used [6]. Runge–Kutta methods are often preferred, since Taylor methods require higher-order derivatives of the function f . However, with the development of computer algebra systems these derivatives can be computed automatically for a wide class of functions [8], and the Taylor approach has regained its attractiveness.

There is one more reason to turn to Taylor methods. Since the second half of the 20th century, numerical computations have been used not only to approximate trajectories of dynamical systems but also to prove theorems about their properties [1, 10]. Such applications require rigorous deductive proofs of inequalities of the form (2) with explicitly known constants, and from this point of view Taylor schemes are more convenient than Runge–Kutta schemes, because their error is expressed directly through the derivatives of the solution.

In [3] and [4] (see also [2]), inequality (2) was proved with explicit constants for schemes of order 3 and 4. As the order grows, the number of terms in the expansions increases rapidly, and a direct extension of that technique becomes cumbersome. In the present paper we treat the case $s = 5$ for quadratic systems, a class that includes many basic models (the Lorenz and Rössler systems, Lotka–Volterra models, etc.). For a quadratic mapping the Taylor formula is finite; this allows us to compute the residual of the approximate solution on each step exactly, as a polynomial in the local time, and to estimate its coefficient by coefficient. As a result we obtain explicit values of the constants in (2), and the bounds for the coefficients of the residual turn out to be sharp.

2. THE PROBLEM OF ESTIMATING THE ERROR OF THE APPROXIMATE SOLUTION

The accuracy of approximate solutions is discussed with various degrees of detail in the literature on numerical methods. The monographs [7] and [12] concentrate mainly on Runge–Kutta methods. Part III of Bakhvalov's monograph [5] deals with methods for ordinary differential equations, and its §1 analyses the Taylor method from a methodological point of view. Part II, §2 of [13] also touches upon the Taylor method: it is noted there that if the first n terms of the Taylor formula are used, then the error is

$$\max_{0 \leq n \leq N} |x_n - y(x_n)| = O(h^n),$$

and the discussion ends with this remark.

In this paper the upper bound of the error is written explicitly, and inequality (2) is proved rigorously with explicit constants. Estimates of this kind are needed when mathematical theorems are proved on the basis of numerical solutions. In recent decades, computational methods have been widely used to obtain rigorous results about dynamical systems, and some long-standing problems have been solved in this way. In particular, in Tucker's works [14, 15] numerical and analytical approaches were combined to prove the existence of the Lorenz attractor. Various methods have also been developed for establishing the existence of homoclinic trajectories and for studying other properties of dynamical systems; see [9, 10].

3. CONSTRUCTION OF AN APPROXIMATE SOLUTION SCHEME BASED ON THE TAYLOR FORMULA

We call system (1) *quadratic* if every component of f is a polynomial of degree at most two in x . Then the second derivative f'' does not depend on x , all derivatives of order three and higher vanish, and for all x, p

$$f(x+p) = f(x) + f'(x)p + 1/2 f''(p, p). \quad (3)$$

Since a quadratic function $f(x)$ is locally Lipschitz, the solution of problem (1) is unique; however, estimating its interval of existence is a rather difficult problem [11]. Therefore we begin with the following condition.

Assumption 1. The solution $x_*(t)$ of the Cauchy problem (1) exists on the given segment $[0; T]$.

We restrict ourselves to the uniform grid $t_n = nh$, where $h = T/N$, $N \in \mathbb{N}$. The Taylor formula for the exact solution with accuracy $O(h^5)$ reads

$$\begin{aligned} x_*(t+h) &= \\ &= x_*(t) + h\dot{x}_*(t) + \frac{h^2}{2!}\ddot{x}_*(t) + \frac{h^3}{3!}\ddot{\ddot{x}}_*(t) + \\ &+ \frac{h^4}{4!}x_*^{(IV)}(t) + \frac{h^5}{5!}x_*^{(V)}(t) + R_6(t, h). \end{aligned}$$

As in [3, 4], we interpret expressions involving f and its derivatives as operators, e.g.

$$\begin{aligned} f'f'f &= f'(x)(f'(x)f(x)), \\ f''(f'f)f &= f''(f'(x)f(x), f(x)), \end{aligned}$$

and introduce the notation

$$F_0 = f, \quad F_{k+1} = F'_k f, \quad k = 0, 1, 2, \dots$$

Then $x_*^{(k+1)}(t) = F_k[x_*(t)]$. Differentiating along the solution and using $f''' \equiv 0$, we obtain for quadratic systems

$$\begin{aligned} F_1 &= f'f, \quad F_2 = f''ff + f'f'f, \\ F_3 &= 3f''(f'f)f + f'f''ff + f'f'f'f, \\ F_4 &= 4f''(f''ff)f + \\ &+ 4f''(f'f'f)f + 3f''(f'f)(f'f) + \\ &+ 3f'f''(f'f)f + f'f'f''ff + f'f'f'f'f. \end{aligned}$$

Indeed, along a solution $\frac{d}{dt}f = f'f$, $\frac{d}{dt}(f'u) = f''fu + f'u$ and $\frac{d}{dt}f''uv = f''\dot{u}v + f''u\dot{v}$, since f'' is constant; applying these rules successively to F_1, F_2, F_3 and using the symmetry of f'' , we get the formulas above. Hence the Taylor expansion takes the form

$$\begin{aligned} &x_*((n+1)h) \\ &= x_*(nh) + \sum_{k=1}^5 \frac{h^k}{k!} F_{k-1}[x_*(nh)] + R_6(nh, h). \end{aligned}$$

Neglecting the remainder term, we arrive at the recurrence formula

$$x_{n+1} = x_n + \sum_{k=1}^5 \frac{h^k}{k!} F_{k-1}(x_n),$$

and x_n is regarded as an approximation of $x_*(nh)$. In theoretical problems the values of the solution between the nodes are also needed, so we use the following continuous approximate solution.

Definition 1. The function

$$\tilde{x}(t) = x_0 + \int_0^t \sum_{k=0}^4 \frac{(s-\sigma_s)^k}{k!} F_k[\tilde{x}(\sigma_s)] ds \quad (4)$$

is called an approximate solution of problem (1), where $\sigma_s = h[s/h]$ is the step function of the grid. It has the properties: $nh \leq \sigma_t < (n+1)h$ for $n = [t/h]$; $\sigma(\sigma_t) = \sigma_t$; $\sigma_s = \sigma_t$ if s and t lie in the same grid interval.

Although (4) has the form of an integral equation, it is in fact a recurrence formula that defines $\tilde{x}(t)$ explicitly: on $[nh, (n+1)h]$

$$\tilde{x}(t) = x_n + \sum_{k=1}^5 \frac{(t-nh)^k}{k!} F_{k-1}(x_n).$$

Lemma 1. For every $n \geq 1$ the equality $\tilde{x}(nh) = x_n$ holds.

The lemma is proved by induction on the segments $[0, nh]$.

Assumption 2. There exists a compact convex set $K \subset \mathbb{R}^d$ such that $x_*(t) \in K$ and $\tilde{x}(t) \in K$ for all $t \in [0, T]$.

We introduce the constants

$$M_0 = \max_{x \in K} |f(x)|, \quad M_1 = \max_{x \in K} \|f'(x)\|, \\ M_2 = \max_{x \in K} \|f''(x)\|, \quad M_3 = 0,$$

where the last equality holds because the system is quadratic; we assume $M_1 > 0$. From the formulas for F_k it follows that for $x \in K$

$$|F_0| \leq M_0, \quad |F_1| \leq M_1 M_0, \\ |F_2| \leq M_2 M_0^2 + M_1^2 M_0, \\ |F_3| \leq 4M_1 M_2 M_0^2 + M_1^3 M_0, \\ |F_4| \leq M_1^4 M_0 + 11M_1^2 M_2 M_0^2 + 4M_2^2 M_0^3. \quad (5)$$

4. MAIN RESULT

By (4), for $t \neq nh$ ($n = 1, 2, \dots, N-1$) the derivative of the approximate solution is

$$\dot{\tilde{x}}(t) = \sum_{k=0}^4 \frac{(t-\sigma_t)^k}{k!} F_k[\tilde{x}(\sigma_t)].$$

Consider the residual of the approximate solution in equation (1):

$$I(s) = f[\tilde{x}(s)] - \sum_{k=0}^4 \frac{(s-\sigma_s)^k}{k!} F_k[\tilde{x}(\sigma_s)]. \quad (6)$$

Lemma 2. For $nh \leq s < (n+1)h$

$$I(s) = \sum_{m=5}^{10} \frac{(s-nh)^m}{m!} G_m,$$

where $F_k = F_k(x_n)$, $G_5 = f'F_4 + 5f''F_0F_3 + 10f''F_1F_2 = F_5(x_n)$ and, for $m = 6, \dots, 10$,

$$G_m = \frac{1}{2} \sum_{\substack{i+j=m \\ 1 \leq i, j \leq 5}} \binom{m}{i} f''F_{i-1}F_{j-1}.$$

Proof. On this interval $\sigma_s = nh$ and $\tilde{x}(s) = x_n + p(s)$, where $p(s) = \sum_{k=1}^5 \frac{(s-nh)^k}{k!} F_{k-1}(x_n)$. By (3), $I(s) = f(x_n) + f'(x_n)p(s) + 1/2 f''(p(s), p(s)) - \sum_{k=0}^4 \frac{(s-nh)^k}{k!} F_k(x_n)$ is a polynomial of degree at most 10 in $s - nh$. Since p has degree 5, for $m \geq 6$ the coefficient of $(s - nh)^m$ comes only from $1/2 f''(p, p)$ and equals $1/2 \sum_{i+j=m} f''F_{i-1}F_{j-1}/(i!j!) = G_m/m!$; similarly, the coefficient of $(s - nh)^5$ is $G_5/5!$. Let

$z(s)$ be the solution of (1) with $z(nh) = x_n$. Then $z^{(k+1)}(nh) = F_k(x_n)$ and $z(s) = \tilde{x}(s) + O((s - nh)^6)$, hence

$$f[\tilde{x}(s)] = f[z(s)] + O((s - nh)^6) = \\ = \sum_{k=0}^5 \frac{(s - nh)^k}{k!} F_k(x_n) + O((s - nh)^6).$$

Therefore the terms of degree 0, ..., 4 cancel and $G_5 = F_5(x_n)$. $\square$

Theorem 1. Let system (1) be quadratic and let Assumptions 1 and 2 hold. Then for $0 \leq t \leq T$ the difference between the exact and approximate solutions satisfies

$$|x_*(t) - \tilde{x}(t)| \leq \\ \leq h^5 \left(\sum_{k=0}^5 \frac{A_k h^k}{(k+6)!} \right) \left(\frac{e^{M_1 t} - 1}{M_1} + h e^{M_1 t} \right),$$

where

$$A_0 = M_1^5 M_0 + 26M_1^3 M_2 M_0^2 + 34M_1 M_2^2 M_0^3, \\ A_1 = 31M_1^4 M_2 M_0^2 + 146M_1^2 M_2^2 M_0^3 + 34M_2^3 M_0^4, \\ A_2 = 56M_1^5 M_2 M_0^2 + \\ 406M_1^3 M_2^2 M_0^3 + 224M_1 M_2^3 M_0^4, \\ A_3 = 91M_1^6 M_2 M_0^2 + 952M_1^4 M_2^2 M_0^3 + \\ + 1400M_1^2 M_2^3 M_0^4 + 224M_2^4 M_0^5, \\ A_4 = 126M_1^7 M_2 M_0^2 + 1890M_1^5 M_2^2 M_0^3 + \\ + 6048M_1^3 M_2^3 M_0^4 + 2016M_1 M_2^4 M_0^5, \\ A_5 = 126M_1^8 M_2 M_0^2 + \\ 2772M_1^6 M_2^2 M_0^3 + 16254M_1^4 M_2^3 M_0^4 + \\ + 11088M_1^2 M_2^4 M_0^5 + 2016M_2^5 M_0^6.$$

In particular, inequality (2) holds with $s = 5$, $L = M_1$ and $C = (T + h) \sum_{k=0}^5 A_k h^k / (k + 6)!$.

Proof. For the exact solution we have

$$x_*(t) = x_0 + \int_0^t f[x_*(s)] ds,$$

and for the approximate solution, from formulas (4) and (6), $\tilde{x}(t) = x_0 + \int_0^t (f[\tilde{x}(s)] - I(s)) ds$. Hence

$$|x_*(t) - \tilde{x}(t)| \leq \\ \leq \left| \int_0^t I(s) ds \right| + \int_0^t |f[x_*(s)] - f[\tilde{x}(s)]| ds \quad (7)$$

Substituting the bounds (5), $\|f'\| \leq M_1$ and $|f''uv| \leq M_2|u||v|$ into the formulas for G_m gives $|G_{k+5}| \leq A_k$, $k = 0, \dots, 5$ (for instance, A_0 is obtained from $G_5 = f'F_4 + 5f''F_0F_3 + 10f''F_1F_2$). Therefore, by Lemma 2,

$$|I(s)| \leq \sum_{k=0}^5 \frac{A_k}{(k+5)!} (s - \sigma_s)^{k+5}.$$

Integrating this bound over a grid interval (or a part of it) gives at most $\sum_{k=0}^5 A_k h^{k+6} / (k + 6)!$. The segment $[0, t]$ meets at most $t/h + 1$ grid intervals, so

$$\left| \int_0^t I(s) ds \right| \leq (t+h) h^5 \sum_{k=0}^5 \frac{A_k h^k}{(k+6)!}.$$

By Assumption 2, since K is a compact convex set, the mean value theorem (Hadamard's lemma) gives $|f[x_*(s)] - f[\tilde{x}(s)]| \leq M_1 |x_*(s) - \tilde{x}(s)|$. Therefore

$$|x_*(t) - \tilde{x}(t)| \leq (t+h) h^5 \sum_{k=0}^5 \frac{A_k h^k}{(k+6)!} + M_1 \int_0^t |x_*(s) - \tilde{x}(s)| ds. \quad (8)$$

We apply the Gronwall lemma in the following form [11]: if $u(t) \geq 0$ is continuous on $[0, T]$, $\alpha, \beta, L \geq 0$ and $u(t) \leq \alpha + \beta t + L \int_0^t u(s) ds$, then

$$u(t) \leq \alpha e^{Lt} + \beta \frac{e^{Lt} - 1}{L}.$$

Applying it to (8) with $L = M_1$, $\alpha = h\beta$ and $\beta = h^5 \sum_{k=0}^5 A_k h^k / (k+6)!$, we obtain the required inequality. This completes the proof of the theorem. $\square$

Remark 1. Estimating $|I(s)|$ pointwise by $\sum_k A_k h^{k+5} / (k+5)!$ would lead to the factors $1/(k+5)!$ instead of $1/(k+6)!$, i.e. to a six times larger leading term. The denominator $6!$ is natural: the local error of the scheme on one step equals $F_5(x_n)h^6/6! + O(h^7)$.

Remark 2. For the scalar equation $\dot{x} = x^2$ and $K = [0, \rho]$ we have $F_k(x) = (k+1)! x^{k+2}$, $M_0 = \rho^2$, $M_1 = 2\rho$, $M_2 = 2$. A direct substitution shows that at $x_n = \rho$ the bounds (5) become equalities and $I(s) = \sum_{m=5}^{10} (11-m) \rho^{m+2} (s-nh)^m$, while $A_{m-5}/m! = (11-m)\rho^{m+2}$. Hence the bounds $|G_{k+5}| \leq A_k$ are attained, and the constants A_k cannot be decreased without additional assumptions on the system.

References

- [1] Azamov A.A., Akhmedov O.S. Existence of a complex closed trajectory in a three-dimensional dynamical system. *Comput. Math. and Math. Phys.* 51, 1353–1359 (2011).
- [2] Azamov A.A., Abdullaev A.Kh., Tilavov A.M. On the derivation of an inequality for estimating the accuracy of an approximate solution of the Cauchy problem. *Bulletin of the Institute of Mathematics*, 2022, Vol. 5, No. 5, pp. 105–111 (in Russian).
- [3] Azamov A., Abdullayev A., Ruziboev M. An explicit estimate for approximate solutions of ODEs based on the Taylor formula. *Ural Mathematical Journal*, Vol. 10, No. 1, 2024, pp. 18–27.
- [4] Abdullaev A., Azamov A., Ruziboev M., Tilavov A. On the accuracy of approximate solution scheme based on the Taylor formula for a Cauchy problem for ODEs. *Uzbek Mathematical Journal* 2026, Volume 70, Issue 2, pp. 5–14. DOI: 10.29229/uzmj.2026-2-1.

- [5] Bakhvalov N.S. *Numerical Methods*. Moscow, Nauka (1973).
- [6] Berezin I.S., Zhidkov N.P. *Metody vychisleniy*, Vol. 2. Moscow (1959) (in Russian).
- [7] Butcher J.C. *Numerical Methods for Ordinary Differential Equations*. John Wiley & Sons, Ltd., United Kingdom (2016).
- [8] Von zur Gathen J., Gerhard J. *Modern Computer Algebra*. Cambridge University Press, 2013. ISBN 978-1-107-03903-2.
- [9] Van den Berg J.B., Mireles-James J.D., Lessard J.-P., Mischaikow K. Rigorous numerics for symmetric connecting orbits: even homoclinics of the Gray–Scott equation. *SIAM J. Math. Anal.* 43 (2011), no. 4, 1557–1594.
- [10] Gameiro M., Lessard J.-P. Rigorous computation of smooth branches of equilibria for the three dimensional Cahn–Hilliard equation. *Numer. Math.* 117 (2011), no. 4, 753–778.
- [11] Hartman P. *Ordinary Differential Equations*. SIAM, XVIII+624 p. (2002).
- [12] Hairer E., Nørsett S., Wanner G. *Solving Ordinary Differential Equations I. Nonstiff Problems*. 2nd edition, Springer (2008).
- [13] Krylov V.I., Bobkov V.V., Monastyrnyi P.I. *Vychislitelnye metody*, Vol. II. Moscow, Nauka (1977) (in Russian).
- [14] Tucker W. The Lorenz attractor exists. *C. R. Acad. Sci. Paris, Série I, Mathématique*, Vol. 328, 12 (1999), 1197–1202.
- [15] Tucker W. A rigorous ODE solver and Smale's 14th problem. *Found. Comput. Math.* 2, 53–117 (2002).

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Краевая задача с условиями неймана для уравнения эллиптического типа с тремя сингулярными коэффициентами в первом октанте

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Аннотация. В работе исследована краевая задача типа Дирихле–Неймана для трёхмерного эллиптического уравнения с тремя сингулярными коэффициентами в первом октанте. Доказана однозначная разрешимость поставленной задачи в классе регулярных решений. Методом интегралов энергии доказана единственность решения поставленной задачи. Для доказательства существования решения использован метод интегрального преобразования Ханкеля. Интегральное преобразование Ханкеля целесообразно применять, очевидно, в случае, когда переменные в уравнении изменяются от 0 до бесконечности; поставленная задача является соответствующим примером. Данное преобразование является одним из эффективных методов нахождения решений таких задач. Для вывода уравнения относительно образа в трёхмерном пространстве интегральное преобразование Ханкеля применяется к исходному уравнению по переменным x и y . В результате возникает краевая задача для обыкновенного дифференциального уравнения по переменной z . С помощью решения этой задачи построено решение поставленной задачи в виде двукратного несобственного интеграла, содержащего функции Бесселя первого рода и функции Макдональда. При обосновании равномерной сходимости несобственных интегралов использовались асимптотические оценки функций Бесселя первого рода и функции Макдональда. На их основе получены оценки для подынтегральных выражений, что позволило доказать сходимость полученного двукратного несобственного интеграла, т. е. сходимость решения поставленной задачи и его производных до второго порядка включительно, а также доказать теорему существования решения в классе регулярных решений.

Ключевые слова: интегральное преобразование Ханкеля, функция Бесселя, функция Бесселя мнимого аргумента, функция Макдональда, сингулярные коэффициенты, уравнения эллиптического типа.

Abstract. A Dirichlet–Neumann type boundary value problem for a three-dimensional elliptic equation with three singular coefficients in the first octant is studied. The unique solvability of the problem in the class of regular solutions is proved. The uniqueness of the solution is proved by the energy integral method. To prove the existence of a solution, the Hankel integral transform method is used. The Hankel integral transform is naturally applied when the variables of the equation range from 0 to infinity, and the problem under consideration is an example of this case. This transform is one of the effective methods for finding solutions of such problems. To derive an equation for the image in three-dimensional space, the Hankel integral transform is applied to the original equation with respect to the variables x and y . As a result, a boundary value problem for an ordinary differential equation in the variable z arises. Using the solution of this problem, the solution of the original problem is constructed in the form of a double improper integral containing Bessel functions of the first kind and Macdonald functions. To justify the uniform convergence of the improper integrals, asymptotic estimates of the Bessel functions of the first kind and of the Macdonald function are used. On this basis, estimates for the integrands are obtained, which makes it possible to prove the convergence of the resulting double improper integral, i.e. the convergence of the solution of the problem and its derivatives up to the second order inclusive, and to prove the existence theorem in the class of regular solutions.

Keywords: Hankel integral transform, Bessel function, modified Bessel function, Macdonald function, singular coefficients, elliptic equations.

Annotatsiya. Ishda birinchi oktantda uchta singulyar koeffitsiyentli uch o'lovli elliptik tenglama uchun Dirixle–Neyman tipidagi chegaraviy masala tadqiq qilingan. Qo'yilgan masalaning reguliyar yechimlar sinfidagi bir qiymatli yechilishi isbotlangan. Energiya integrallari usuli bilan masala yechimining yagonaligi isbotlangan. Yechimning mavjudligini isbotlash uchun Hankel integral almashtirishi usulidan foydalanilgan. Hankel integral almashtirishini tenglamadagi o'zgaruvchilar 0 dan cheksizgacha o'zgargan holda qo'llash maqsadga muvofiq bo'lib, qo'yilgan masala bunga mos misol bo'ladi. Ushbu almashtirish bunday masalalarning yechimlarini topishning samarali usullaridan biridir. Uch o'lovli fazoda tasvirga nisbatan tenglamani keltirib chiqarish uchun Hankel integral almashtirishi dastlabki tenglamaga x va y o'zgaruvchilar bo'yicha qo'llaniladi. Natijada z o'zgaruvchi bo'yicha oddiy differensial tenglama uchun chegaraviy masala hosil bo'ladi. Bu masalaning yechimi yordamida qo'yilgan masalaning yechimi birinchi tur Bessel funksiyalari va Makdonald funksiyalarini o'z ichiga olgan ikki karrali xosmas integral ko'rinishida qurilgan. Xosmas integrallarning tekis yaqinlashishini asoslashda birinchi tur Bessel funksiyalari va Makdonald funksiyasining asimptotik baholaridan foydalanilgan. Ular asosida integral ostidagi ifodalar uchun baholar olingan, bu esa hosil qilingan ikki karrali xosmas integralning yaqinlashishini, ya'ni qo'yilgan masala yechimi va uning ikkinchi tartibgacha (ikkinchi tartib ham kiradi) hosilalarining yaqinlashishini, shuningdek, reguliyar yechimlar sinfidagi yechimning mavjudlik teoremasini isbotlash imkonini berdi.

Kalit so'zlar: Hankel integral almashtirishi, Bessel funksiyasi, mavhum argumentli Bessel funksiyasi, Makdonald funksiyasi, singulyar koeffitsiyentlar, elliptik tipdagi tenglamalar.

1. ВВЕДЕНИЕ. ПОСТАНОВКА ЗАДАЧИ

Рассмотрим уравнение

$$\left(B_{\alpha-1/2}^x + B_{\beta-1/2}^y + B_{\gamma-1/2}^z\right)u(x, y, z) = 0, \\ (x, y, z) \in \Omega \quad (1)$$

в области $\Omega = \{(x, y, z): x \in (0; +\infty), y \in (0; +\infty), z \in (0; +\infty)\}$, где $u(x, y, z)$ — неизвестная функция, а $\alpha, \beta, \gamma \in \mathbb{R}$, причём $0 < \alpha, \beta, \gamma < 1/2$.

Для уравнения (1) в области Ω сформулируем и исследуем следующую задачу.

Задача DN_∞ . Найти решение уравнения (1) в области Ω , удовлетворяющее условиям

$$u \in C(\overline{\Omega}),$$

$$B_{\alpha-1/2}^x u, B_{\beta-1/2}^y u, B_{\gamma-1/2}^z u \in C(\Omega), \quad (2)$$

$$\lim_{x \rightarrow 0} x^{2\alpha} u_x = v_1(y, z),$$

$$\lim_{x \rightarrow \infty} u(x, y, z) = 0, \quad y, z \in [0, +\infty), \quad (3)$$

$$\lim_{y \rightarrow 0} y^{2\beta} u_y = v_2(x, z),$$

$$\lim_{y \rightarrow \infty} u(x, y, z) = 0, \quad x, z \in [0, +\infty), \quad (4)$$

$$\lim_{z \rightarrow 0} z^{2\gamma} u_z = v_3(x, y),$$

$$\lim_{z \rightarrow \infty} u(x, y, z) = 0, \quad x, y \in [0, +\infty), \quad (5)$$

где $\overline{\Omega} = \{(x, y, z): x \in [0; +\infty), y \in [0; +\infty), z \in [0; +\infty)\}$, а $v_i, i = 1, 2, 3$ — заданные непрерывные функции.

2. ЕДИНСТВЕННОСТЬ РЕШЕНИЯ ЗАДАЧИ DN_∞

Теорема 1. Если существует решение задачи DN_∞ , то оно единственно.

Доказательство. Пусть задача DN_∞ имеет два решения $u_1(x, y, z)$ и $u_2(x, y, z)$. Тогда $u(x, y, z) = u_1(x, y, z) - u_2(x, y, z)$ удовлетворяет уравнению (1) и однородным краевым условиям. Докажем, что $u(x, y, z) \equiv 0$ в $\overline{\Omega}$.

$$\begin{aligned} & \text{В области } \Omega \text{ справедливо тождество} \\ & x^{2\alpha} y^{2\beta} z^{2\gamma} u Lu = (x^{2\alpha} y^{2\beta} z^{2\gamma} u u_x)_x + \\ & + (x^{2\alpha} y^{2\beta} z^{2\gamma} u u_y)_y + \\ & + (x^{2\alpha} y^{2\beta} z^{2\gamma} u u_z)_z - \\ & - x^{2\alpha} y^{2\beta} z^{2\gamma} (u_x^2 + u_y^2 + u_z^2) = 0. \end{aligned}$$

Интегрируя это тождество по области

$$\Omega_{\delta_1 \delta_3 \delta_5}^{\delta_2 \delta_4 \delta_6} = \{(x, y, z): \delta_1 < x < \delta_2, \\ \delta_3 < y < \delta_4, \delta_5 < z < \delta_6\},$$

где $\delta_j, j = \overline{1, 6}$ — положительные числа, имеем

$$\begin{aligned} & \iiint_{\Omega_{\delta_1 \delta_3 \delta_5}^{\delta_2 \delta_4 \delta_6}} [(x^{2\alpha} y^{2\beta} z^{2\gamma} u u_x)_x + \\ & + (x^{2\alpha} y^{2\beta} z^{2\gamma} u u_y)_y + \\ & + (x^{2\alpha} y^{2\beta} z^{2\gamma} u u_z)_z] dx dy dz = \\ & = \iiint_{\Omega_{\delta_1 \delta_3 \delta_5}^{\delta_2 \delta_4 \delta_6}} [x^{2\alpha} y^{2\beta} z^{2\gamma} \times \end{aligned}$$

$$\times (u_x^2 + u_y^2 + u_z^2)] dx dy dz. \quad (6)$$

Очевидно, что если $\delta_1, \delta_3, \delta_5 \rightarrow 0, \delta_2, \delta_4, \delta_6 \rightarrow +\infty$, то $\Omega_{\delta_1 \delta_3 \delta_5}^{\delta_2 \delta_4 \delta_6} \rightarrow \Omega$.

Применяя формулу Гаусса–Остроградского [15] к левой стороне равенства (6), имеем

$$\begin{aligned} & \int_{\delta_5}^{\delta_6} \int_{\delta_3}^{\delta_4} y^{2\beta} z^{2\gamma} [\delta_2^{2\alpha} u_x(\delta_2, y, z) u(\delta_2, y, z) - \\ & - \delta_1^{2\alpha} u_x(\delta_1, y, z) u(\delta_1, y, z)] dy dz + \\ & + \int_{\delta_5}^{\delta_6} \int_{\delta_1}^{\delta_2} x^{2\alpha} z^{2\gamma} [\delta_4^{2\beta} u_y(x, \delta_4, z) u(x, \delta_4, z) - \\ & - \delta_3^{2\beta} u_y(x, \delta_3, z) u(x, \delta_3, z)] dx dz + \\ & + \int_{\delta_3}^{\delta_4} \int_{\delta_1}^{\delta_2} x^{2\alpha} y^{2\beta} [\delta_6^{2\gamma} u_z(x, y, \delta_6) u(x, y, \delta_6) - \\ & - \delta_5^{2\gamma} u_z(x, y, \delta_5) u(x, y, \delta_5)] dx dy = \\ & = \iiint_{\Omega_{\delta_1 \delta_3 \delta_5}^{\delta_2 \delta_4 \delta_6}} [x^{2\alpha} y^{2\beta} z^{2\gamma} (u_x^2 + u_y^2 + u_z^2)] dx dy dz. \end{aligned}$$

Переходя к пределу при $\delta_1, \delta_3, \delta_5 \rightarrow 0, \delta_2, \delta_4, \delta_6 \rightarrow +\infty$ и учитывая условия (2)–(5) (при $v_3(x, y) = 0$), из последнего равенства получим

$$\iiint_{\Omega} [x^{2\alpha} y^{2\beta} z^{2\gamma} (u_x^2 + u_y^2 + u_z^2)] dx dy dz = 0.$$

Следовательно,

$$u_x(x, y, z) \equiv u_y(x, y, z) \equiv u_z(x, y, z) \equiv 0, \\ (x, y, z) \in \Omega.$$

Тогда $u(x, y, z) \equiv \text{const}, (x, y, z) \in \Omega$. Так как $u(x, y, z) \in C(\overline{\Omega})$ и $\lim_{x \rightarrow \infty} u(x, y, z) \equiv 0$, то $u(x, y, z) \equiv 0, (x, y, z) \in \overline{\Omega}$. Отсюда следует утверждение теоремы. $\square$

3. СУЩЕСТВОВАНИЕ РЕШЕНИЯ ЗАДАЧИ DN_∞

В данной работе методом преобразования Ханкеля исследуем поставленную задачу DN_∞ . Многие задачи физики, прикладной математики и математического моделирования сводятся к решению дифференциальных, интегральных и интегро-дифференциальных уравнений. Одним из эффективных методов нахождения аналитического решения является метод интегральных преобразований. При этом наиболее изученным и часто применяемым из всех преобразований Бесселя является интегральное преобразование Ханкеля.

Интегральным преобразованием Ханкеля порядка ν функции $f(t)$ называется интеграл [11], [12]

$$\bar{f}(p) = \int_0^{+\infty} f(t) t J_\nu(pt) dt,$$

$$\nu \geq -1/2, \quad 0 < p < +\infty. \quad (7)$$

Здесь $J_\nu(z)$ — функция Бесселя первого рода порядка ν [13].

Преобразование Ханкеля функции $f(t)$ верно для любых точек на интервале $(0, +\infty)$, в которых функция $f(t)$ непрерывна или кусочно-непрерывна с конечным числом точек разрыва первого рода, и

$$\int_0^{+\infty} |f(t)| t^{1/2} dt < +\infty.$$

Формула обращения преобразования Ханкеля определяется интегралом

$$f(t) = \int_0^{+\infty} \bar{f}(p) p J_\nu(pt) dp, \quad 0 < t < +\infty. \quad (8)$$

Функцию $\bar{f}(p)$ часто называют образом Фурье–Бесселя–Ханкеля [14], а функцию $f(t)$ — оригиналом.

Преобразование Ханкеля целесообразно применять, очевидно, в случае, когда в уравнении переменные изменяются от 0 до $+\infty$. Поставленная задача DN_∞ является соответствующим примером.

Для вывода уравнения образа $\tilde{u}_{\lambda\mu}(z)$ применим преобразование Ханкеля к исходному уравнению (1) по переменным x и y . Имеем

$$\begin{aligned} \tilde{u}_{\lambda\mu}(z) = & \int_0^{+\infty} \int_0^{+\infty} x y [x^{\alpha-1/2} y^{\beta-1/2} u(x, y, z)] \times \\ & \times J_{1/2-\alpha}(\lambda x) J_{1/2-\beta}(\mu y) dx dy, \end{aligned} \quad (9)$$

причём

$$\begin{aligned} u(x, y, z) = & x^{1/2-\alpha} y^{1/2-\beta} \times \\ & \times \int_0^{+\infty} \int_0^{+\infty} \lambda \mu \tilde{u}_{\lambda\mu}(z) \times \\ & \times J_{1/2-\alpha}(\lambda x) J_{1/2-\beta}(\mu y) d\lambda d\mu. \end{aligned} \quad (10)$$

На основании (9) введём функции

$$\begin{aligned} \tilde{u}_{\varepsilon_1 \varepsilon_3 \lambda \mu}^{\varepsilon_2 \varepsilon_4}(z) = & \int_{\varepsilon_3}^{\varepsilon_4} \int_{\varepsilon_1}^{\varepsilon_2} x^{1/2+\alpha} y^{1/2+\beta} u(x, y, z) \times \\ & \times J_{1/2-\alpha}(\lambda x) J_{1/2-\beta}(\mu y) dx dy, \end{aligned} \quad (11)$$

где $\varepsilon_j, j = \overline{1,4}$ — положительные числа.

Очевидно, что

$$\lim_{\substack{\varepsilon_1, \varepsilon_3 \rightarrow 0 \\ \varepsilon_2, \varepsilon_4 \rightarrow +\infty}} \tilde{u}_{\varepsilon_1 \varepsilon_3 \lambda \mu}^{\varepsilon_2 \varepsilon_4}(z) = \tilde{u}_{\lambda\mu}(z).$$

С помощью функции (11) и уравнения (1) вычислим выражение $B_{\gamma-1/2}^z \tilde{u}_{\varepsilon_1 \varepsilon_3 \lambda \mu}^{\varepsilon_2 \varepsilon_4}(z)$:

$$B_{\gamma-1/2}^z \tilde{u}_{\varepsilon_1 \varepsilon_3}^{\varepsilon_2 \varepsilon_4} = \int_{\varepsilon_3}^{\varepsilon_4} \int_{\varepsilon_1}^{\varepsilon_2} x^{1/2+\alpha} y^{1/2+\beta} \times$$

$$\begin{aligned} & \times J_{1/2-\alpha}(\lambda x) J_{1/2-\beta}(\mu y) \times \\ & \times B_{\gamma-1/2}^z u(x, y, z) dx dy = \\ & = - \int_{\varepsilon_3}^{\varepsilon_4} \int_{\varepsilon_1}^{\varepsilon_2} x^{1/2+\alpha} y^{1/2+\beta} \times \\ & \times J_{1/2-\alpha}(\lambda x) J_{1/2-\beta}(\mu y) \times \\ & \times (B_{\alpha-1/2}^x + B_{\beta-1/2}^y) u(x, y, z) dx dy = \\ & = - \int_{\varepsilon_3}^{\varepsilon_4} \left[\int_{\varepsilon_1}^{\varepsilon_2} x^{1/2+\alpha} J_{1/2-\alpha}(\lambda x) \times \right. \\ & \times B_{\alpha-1/2}^x u(x, y, z) dx \times \\ & \times y^{1/2+\beta} J_{1/2-\beta}(\mu y) dy - \\ & \left. - \int_{\varepsilon_1}^{\varepsilon_2} \left[\int_{\varepsilon_3}^{\varepsilon_4} y^{1/2+\beta} J_{1/2-\beta}(\mu y) \times \right. \right. \\ & \times B_{\beta-1/2}^y u(x, y, z) dy \times \\ & \times x^{1/2+\alpha} J_{1/2-\alpha}(\lambda x) dx, \end{aligned} \quad (12)$$

где $B_q^z \equiv \frac{\partial^2}{\partial z^2} + \frac{2q+1}{z} \frac{\partial}{\partial z}$ — оператор Бесселя [16].

Применяя правило интегрирования по частям, из (12) получим

$$\begin{aligned} B_{\gamma-1/2}^z \tilde{u}_{\varepsilon_1 \varepsilon_3 \lambda \mu}^{\varepsilon_2 \varepsilon_4}(z) = & - \int_{\varepsilon_3}^{\varepsilon_4} \{ [J_{1/2-\alpha}(\lambda x) u_x - \\ & - \lambda J_{-1/2-\alpha}(\lambda x) u] x^{1/2+\alpha} \Big|_{x=\varepsilon_1}^{x=\varepsilon_2} - \\ & - \lambda^2 \int_{\varepsilon_1}^{\varepsilon_2} u(x, y, z) x^{1/2+\alpha} J_{1/2-\alpha}(\lambda x) dx \} \times \\ & \times y^{1/2+\beta} J_{1/2-\beta}(\mu y) dy - \\ & - \int_{\varepsilon_1}^{\varepsilon_2} \{ [J_{1/2-\beta}(\mu y) u_y - \\ & - \mu J_{-1/2-\beta}(\mu y) u] y^{1/2+\beta} \Big|_{y=\varepsilon_3}^{y=\varepsilon_4} - \\ & - \mu^2 \int_{\varepsilon_3}^{\varepsilon_4} u(x, y, z) y^{1/2+\beta} J_{1/2-\beta}(\mu y) dy \} \times \\ & \times x^{1/2+\alpha} J_{1/2-\alpha}(\lambda x) dx. \end{aligned} \quad (13)$$

Непосредственным вычислением можно убедиться, что при фиксированных $\lambda \in (0, +\infty)$ и $\mu \in (0, +\infty)$ следующие пределы существуют и конечны:

$$\begin{aligned} \lim_{x \rightarrow 0} x^{1/2+\alpha} J_{-1/2-\alpha}(\lambda x) = & 2^{1/2+\alpha} \lambda^{-1/2-\alpha} / \Gamma(1/2 - \alpha), \end{aligned} \quad (14)$$

$$\begin{aligned} \lim_{y \rightarrow 0} y^{1/2+\beta} J_{-1/2-\beta}(\mu y) = & 2^{1/2+\beta} \mu^{-1/2-\beta} / \Gamma(1/2 - \beta). \end{aligned} \quad (15)$$

Поведение функции $J_\nu(x)$ при малых и больших значениях x даётся соответственно формулами [18]:

$$J_\nu(x) \underset{x \rightarrow 0}{\approx} \frac{x^\nu}{2^\nu \Gamma(1 + \nu)},$$

$$J_\nu(x) \underset{x \rightarrow +\infty}{\approx} \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right). \quad (16)$$

Из равенства (13), переходя к пределу при $\varepsilon_1 \rightarrow 0$, $\varepsilon_3 \rightarrow 0$, $\varepsilon_2 \rightarrow +\infty$, $\varepsilon_4 \rightarrow +\infty$ и учитывая условия (2), (3), (4), (5) и равенства (14), (15), а также обозначение (9), получим уравнение вида

$$\tilde{u}_{zz\lambda\mu}(z) + \frac{2\gamma}{z} \tilde{u}_{z\lambda\mu}(z) - \chi^2 \tilde{u}_{\lambda\mu}(z) = 0, \quad 0 < \lambda, \mu < +\infty, \quad 0 < z < c, \quad (17)$$

где $\chi^2 = \lambda^2 + \mu^2$.

Кроме того, в силу граничных условий (5), из (9) следует, что функция $\tilde{u}_{\lambda\mu}(z)$ удовлетворяет следующим граничным условиям:

$$\lim_{z \rightarrow 0} z^{2\gamma} \tilde{u}_{z\lambda\mu}(z) = v_{3\lambda\mu}, \quad \lim_{z \rightarrow \infty} u_{\lambda\mu}(x, y, z) = 0, \quad (18)$$

где

$$v_{3\lambda\mu} = \int_0^{+\infty} \int_0^{+\infty} x^{1/2+\alpha} y^{1/2+\beta} v_3(x, y) \times J_{1/2-\alpha}(\lambda x) J_{1/2-\beta}(\mu y) dx dy. \quad (19)$$

Теперь найдём общее решение уравнения (17) в виде [6]

$$\tilde{u}_{\lambda\mu}(z) = c_1 z^{1/2-\gamma} I_{1/2-\gamma}(\chi z) + c_2 z^{1/2-\gamma} K_{1/2-\gamma}(\chi z), \quad z \in [0, c], \quad (20)$$

где c_1 и c_2 — произвольные постоянные, а $I_l(x)$ и $K_l(x)$ — функция Инфельда и функция Макдональда порядка l [13] соответственно.

Из равенства (18), на основании асимптотических поведений функций $I_\nu(x)$ и $K_\nu(x)$ при больших x [18, с. 172–173]:

$$I_\nu(x) \approx \frac{e^x}{(2\pi x)^{1/2}}, \quad K_\nu(x) \approx \left(\frac{\pi}{2x}\right)^{1/2} e^{-x}, \quad (21)$$

следует, что решение уравнения (17), удовлетворяющее второму условию (18), определяется равенством

$$\tilde{u}_{\lambda\mu}(z) = c_2 z^{1/2-\gamma} K_{1/2-\gamma}(\chi z). \quad (22)$$

Удовлетворяя функцию (20) первому из условий (18), получим

$$\lim_{z \rightarrow 0} z^{2\gamma} \tilde{u}_{z\lambda\mu}(z) = -\lim_{z \rightarrow 0} \chi c_2 z^{1/2+\gamma} K_{-1/2-\gamma}(\chi z) = v_{3\lambda\mu},$$

откуда однозначно находим c_2 :

$$c_2 = v_{3\lambda\mu} \left(\frac{2}{\chi}\right)^{1/2-\gamma} \frac{1}{\Gamma(1/2+\gamma)}.$$

Подставляя значение c_2 в равенство (20) и учитывая $\bar{K}_\nu(x) = 2^{1-\nu} x^\nu K_\nu(x)/\Gamma(\nu)$, $\nu > 0$, однозначно находим решение уравнения (18) в виде

$$\tilde{u}_{\lambda\mu}(z) = v_{3\lambda\mu} \frac{\bar{K}_{1/2-\gamma}(\chi z) \Gamma(1/2-\gamma)}{2^{2\gamma} \chi^{1-2\gamma} \Gamma(1/2+\gamma)}. \quad (23)$$

Решение же исходной задачи получаем с помощью обратного преобразования Ханкеля (10):

$$u(x, y, z) = \int_0^{+\infty} \int_0^{+\infty} \lambda \mu X_\lambda(x) Q_\mu(y) \times \tilde{u}_{\lambda\mu}(z) d\lambda d\mu, \quad (24)$$

где $X_\lambda(x) = x^{1/2-\alpha} J_{1/2-\alpha}(\lambda x)$, $Q_\mu(y) = y^{1/2-\beta} J_{1/2-\beta}(\mu y)$, а $\tilde{u}_{\lambda\mu}(z)$ определяется формулой (23), причём они соответственно являются решениями следующих уравнений:

$$B_{\alpha-1/2}^x X_\lambda(x) = -\lambda^2 X_\lambda(x), \quad 0 < x < +\infty, \quad (25)$$

$$B_{\beta-1/2}^y Q_\mu(y) = -\mu^2 Q_\mu(y), \quad 0 < y < +\infty, \quad (26)$$

$$B_{\gamma-1/2}^z \tilde{u}_{\lambda\mu}(z) = \chi^2 \tilde{u}_{\lambda\mu}(z), \quad \chi^2 = \lambda^2 + \mu^2, \quad 0 < \lambda, \mu, z < +\infty. \quad (27)$$

Если в (24) возможно дифференцирование под знаком интеграла, то функция $u(x, y, z)$ есть решение уравнения (1). Действительно,

$$\begin{aligned} & B_{\alpha-1/2}^x u(x, y, z) + B_{\beta-1/2}^y u(x, y, z) + \\ & + B_{\gamma-1/2}^z u(x, y, z) = \\ & = \int_0^{+\infty} \int_0^{+\infty} \lambda \mu [B_{\alpha-1/2}^x X_\lambda(x)] \times \\ & \times Q_\mu(y) \tilde{u}_{\lambda\mu}(z) d\lambda d\mu + \\ & + \int_0^{+\infty} \int_0^{+\infty} \lambda \mu X_\lambda(x) [B_{\beta-1/2}^y Q_\mu(y)] \times \\ & \times \tilde{u}_{\lambda\mu}(z) d\lambda d\mu + \\ & + \int_0^{+\infty} \int_0^{+\infty} \lambda \mu X_\lambda(x) Q_\mu(y) \times \\ & \times [B_{\gamma-1/2}^z \tilde{u}_{\lambda\mu}(z)] d\lambda d\mu. \end{aligned}$$

Отсюда в силу (25)–(27) имеем

$$B_{\alpha-1/2}^x u(x, y, z) + B_{\beta-1/2}^y u(x, y, z) + B_{\gamma-1/2}^z u(x, y, z) = 0.$$

Покажем, что функция (24) удовлетворяет условиям (4) и (5). Используя формулы (16), функции $X_\lambda(x)$ и $Q_\mu(y)$ при малых и больших значениях аргументов соответственно переписутся в виде

$$X_\lambda(x) \approx \frac{\lambda^{1/2-\alpha} x^{1-2\alpha}}{2^{1/2-\alpha} \Gamma(3/2-\alpha)}, \quad 0 < x, \lambda < 1, \quad (28)$$

$$X_\lambda(x) \approx x^{-\alpha} \left(\frac{2}{\pi\lambda}\right)^{1/2} \sin\left(\lambda x + \frac{\alpha\pi}{2}\right), \quad 1 < x, \lambda < +\infty, \quad (29)$$

$$Q_\mu(y) \approx \frac{\mu^{1/2-\beta} y^{1-2\beta}}{2^{1/2-\beta} \Gamma(3/2-\beta)},$$

$$0 < y, \mu < 1, \quad (30)$$

$$Q_\mu(y) \approx y^{-\beta} \left(\frac{2}{\pi\mu}\right)^{1/2} \sin\left(\mu y + \frac{\beta\pi}{2}\right),$$

$$1 < y, \mu < +\infty. \quad (31)$$

Из приведённых асимптотических соотношений следует, что функция (24) удовлетворяет условиям (4) и (5).

Теперь докажем несколько лемм, которые будут использованы при установлении равномерной сходимости двойного интеграла (24).

Лемма 1. Если $\alpha \in (0, 1/2)$, то относительно функции $X_\lambda(x) = x^{1/2-\alpha} J_{1/2-\alpha}(\lambda x)$ при $x \in [0, +\infty)$ справедливы следующие оценки:

$$|X_\lambda(x)| \leq \begin{cases} c_3 \lambda^{\alpha-1/2}, & 0 < x \cdot \lambda < 1, \\ c_4 \lambda^{\alpha-1/2}, & 1 < x \cdot \lambda < +\infty, \end{cases} \quad (32)$$

$$|x^{2\alpha} X'_\lambda(x)| \leq \begin{cases} c_5 \lambda^{1/2-\alpha}, & 0 < x \cdot \lambda < 1, \\ c_6 \lambda^{1/2-\alpha} (\lambda x)^\alpha, & 1 < x \cdot \lambda < +\infty, \end{cases} \quad (33)$$

$$|B_{\alpha-1/2}^x X_\lambda(x)| \leq \begin{cases} c_7 \lambda^{3/2+\alpha}, & 0 < x \cdot \lambda < 1, \\ c_8 \lambda^{3/2+\alpha}, & 1 < x \cdot \lambda < +\infty, \end{cases} \quad (34)$$

где $c_j, j = \overline{3, 8}$, — положительные постоянные.

Доказательство. Из равенства (25) и определения оператора Бесселя получаем оценку (34).

Теперь рассмотрим функцию $x^{2\alpha} X'_\lambda(x) = \lambda x^{1/2+\alpha} J_{-1/2-\alpha}(\lambda x)$ и равенство (23). В силу асимптотических формул (16), нетрудно показать, что для этих функций справедливы оценки (32) и (33) соответственно. Лемма 1 доказана. $\square$

Аналогично доказывается следующая

Лемма 2. Если $\beta \in (0, 1/2)$, то относительно функции $Q_\mu(y) = y^{1/2-\beta} J_{1/2-\beta}(\mu y)$ при $y \in [0, +\infty)$ справедливы следующие оценки:

$$|Q_\mu(y)| \leq \begin{cases} c_9 \mu^{\beta-1/2}, & 0 < y \cdot \mu < 1, \\ c_{10} \mu^{\beta-1/2}, & 1 < y \cdot \mu < +\infty, \end{cases} \quad (35)$$

$$|y^{2\beta} Q'_\mu(y)| \leq \begin{cases} c_{11} \mu^{1/2-\beta}, & 0 < y \cdot \mu < 1, \\ c_{12} \mu^{1/2-\beta} (\mu y)^\beta, & 1 < y \cdot \mu < +\infty, \end{cases} \quad (36)$$

$$|B_{\beta-1/2}^y Q_\mu(y)| \leq$$

$$\leq \begin{cases} c_{13} \mu^{3/2+\beta}, & 0 < y \cdot \mu < 1, \\ c_{14} \mu^{3/2+\beta}, & 1 < y \cdot \mu < +\infty, \end{cases} \quad (37)$$

где $c_j, j = \overline{9, 14}$, — положительные постоянные.

Лемма 3. При любых $\lambda, \mu \in (0, +\infty)$ и $\forall z \in [0, c]$ для функции $\tilde{y}_{\lambda\mu}(z)$, определённой равенством (23), справедливы оценки

$$|\tilde{y}_{\lambda\mu}(z)| \leq |v_{3\lambda\mu}|,$$

$$|B_{\gamma-1/2}^z \tilde{y}_{\lambda\mu}(z)| \leq \chi^2 |v_{3\lambda\mu}|. \quad (38)$$

Доказательство. Известно [18], что если $v = \text{const} > 0$, то

$$\bar{K}_v(t) \leq 1, \quad \bar{K}_v(0) = 1. \quad (39)$$

Из равенства (23), согласно (39), следует первая оценка из (38).

Как было показано ранее, функция $\tilde{y}_{\lambda\mu}(z)$ удовлетворяет уравнению (27). Отсюда, в силу первой оценки (38), сразу следует справедливость и второй оценки из (38). Лемма 3 доказана. $\square$

Лемма 4. Пусть функция $v_3(x, y)$, $x, y \in (0, 1)$, удовлетворяет следующему условию:

a) $v_3(x, y) \in C(\bar{\Pi})$, где $\Pi = \{(x, y): x \in (0, 1), y \in (0, 1)\}$.

Тогда для коэффициентов

$$v_{3\lambda\mu} = \int_0^1 \int_0^1 x^{1/2+\alpha} y^{1/2+\beta} v_3(x, y) \times$$

$$\times J_{1/2-\alpha}(\lambda x) J_{1/2-\beta}(\mu y) dx dy \quad (40)$$

справедлива следующая оценка:

$$|v_{3\lambda\mu}| \leq C_{19} \lambda^{-1/2-\alpha} \mu^{-1/2-\beta},$$

$$0 < \lambda, \mu < 1.$$

b) Пусть функция $v_3(x, y)$, $x, y \in (1, +\infty)$, удовлетворяет следующим условиям:

I. $v_3(x, y) \in C_{x,y}^{4,4}(\bar{\Pi})$, где $\Pi = \{(x, y): x \in (1, +\infty), y \in (1, +\infty)\}$;

$$\text{II.} \quad \lim_{x \rightarrow 1} \frac{\partial^j}{\partial x^j} v_3(x, y) = 0,$$

$$\lim_{x \rightarrow +\infty} x^\alpha \frac{\partial^j}{\partial x^j} v_3(x, y) = 0, \quad j = \overline{0, 3},$$

$$\int_1^{+\infty} \left| x^\alpha \frac{\partial^4}{\partial x^4} v_3(x, y) \right| dx < +\infty;$$

$$\text{III.} \quad \lim_{y \rightarrow 1} \frac{\partial^j}{\partial y^j} v_3(x, y) = 0,$$

$$\lim_{y \rightarrow +\infty} y^\beta \frac{\partial^j}{\partial y^j} v_3(x, y) = 0, \quad j = \overline{0, 3},$$

$$\int_1^{+\infty} \left| y^\beta \frac{\partial^4}{\partial y^4} v_3(x, y) \right| dy < +\infty.$$

Доказательство. Сначала докажем пункт a) леммы 4.

Рассмотрим коэффициент $v_{3\lambda\mu}$ в промежутке $[0, 1]$. Согласно формуле (40), коэффициенты $v_{3\lambda\mu}$ можно переписать в виде

$$v_{3\lambda\mu} = \int_0^1 y^{1/2+\beta} J_{1/2-\beta}(\mu y) F_\lambda(y) dy, \quad (41)$$

где

$$F_\lambda(y) = \int_0^1 x^{1/2+\alpha} J_{1/2-\alpha}(\lambda x) v_3(x, y) dx.$$

Сначала оценим $F_\lambda(y)$ в промежутке $[0, 1]$:

$$\begin{aligned} |F_\lambda(y)| &\leq \int_0^1 |v_3(x, y)| \cdot \\ &\cdot |x^{1/2+\alpha} J_{1/2-\alpha}(\lambda x)| dx \leq \\ &\leq C_{15} \cdot C_{16} \lambda^{-1/2-\alpha} \leq C_{17} \lambda^{-1/2-\alpha}. \end{aligned}$$

Подставляя найденную оценку $F_\lambda(y)$ в (41), получаем

$$\begin{aligned} |v_{3\lambda\mu}| &\leq \int_0^1 |F_\lambda(y)| \cdot |y^{1/2+\beta} J_{1/2-\beta}(\mu y)| dy \leq \\ &\leq C_{17} \lambda^{-1/2-\alpha} \cdot C_{18} \mu^{-1/2-\beta} = \\ &= C_{19} \lambda^{-1/2-\alpha} \mu^{-1/2-\beta}. \end{aligned}$$

Тем самым доказано утверждение пункта (а) леммы 4. Перейдём к доказательству пункта (б).

Рассмотрим коэффициент $v_{3\lambda\mu}$ в промежутке $[1, +\infty)$. Согласно формуле (19), коэффициенты $v_{3\lambda\mu}$ можно переписать в виде

$$\begin{aligned} v_{3\lambda\mu} &= \int_1^{+\infty} y^{1/2+\beta} J_{1/2-\beta}(\mu y) F_\lambda(y) dy, \\ F_\lambda(y) &= \int_1^{+\infty} x^{1/2+\alpha} J_{1/2-\alpha}(\lambda x) v_3(x, y) dx. \end{aligned}$$

Сначала рассмотрим функцию $F_\lambda(y)$. Пользуясь равенством

$$\begin{aligned} x^{1/2+\alpha} J_{1/2-\alpha}(\lambda x) &= \\ &= -\frac{1}{\lambda} \frac{d}{dx} [x^{1/2+\alpha} J_{-1/2-\alpha}(\lambda x)], \end{aligned}$$

и применяя правило интегрирования по частям четыре раза, из последнего получим

$$\begin{aligned} F_\lambda(y) &= -\frac{1}{\lambda} x^{1/2+\alpha} J_{-1/2-\alpha}(\lambda x) \times \\ &\times v_3(x, y) \Big|_{x=1}^{x=+\infty} - \\ &- \frac{1}{\lambda^2} x^{1/2+\alpha} J_{1/2-\alpha}(\lambda x) v_{3x}(x, y) \Big|_{x=1}^{x=+\infty} - \\ &- \frac{1}{\lambda^3} x^{1/2+\alpha} J_{-1/2-\alpha}(\lambda x) v_{3xx}(x, y) \Big|_{x=1}^{x=+\infty} - \\ &- \frac{1}{\lambda^4} x^{1/2+\alpha} J_{1/2-\alpha}(\lambda x) v_{3xxx}(x, y) \Big|_{x=1}^{x=+\infty} + \\ &+ \frac{1}{\lambda^4} \int_1^{+\infty} v_{3xxxx}(x, y) \times \\ &\times x^{1/2+\alpha} J_{1/2-\alpha}(\lambda x) dx. \end{aligned} \quad (42)$$

В силу условий леммы 4, внеинтегральные слагаемые в (42) равны нулю. Следовательно,

$$\begin{aligned} F_\lambda(y) &= \frac{1}{\lambda^4} \int_1^{+\infty} v_{3xxxx}(x, y) \times \\ &\times x^{1/2+\alpha} J_{1/2-\alpha}(\lambda x) dx. \end{aligned}$$

Теперь рассмотрим коэффициент $v_{3\lambda\mu}$, определяемый равенством (19), в промежутке $[1, +\infty)$. Покажем, что для функции $v_{3\lambda\mu}$ справедлива следующая оценка:

$$\begin{aligned} |v_{3\lambda\mu}| &\leq \frac{1}{\lambda^4 \mu^4} \int_1^{+\infty} \int_1^{+\infty} |x^\alpha y^\beta \frac{\partial^8}{\partial x^4 \partial y^4} v_3(x, y) \\ &\times x^{1/2} J_{1/2-\alpha}(\lambda x) y^{1/2} J_{1/2-\beta}(\mu y)| dx dy \leq \\ &\leq \frac{C_{22}}{\lambda^{4,5} \mu^{4,5}}. \end{aligned}$$

Поскольку функция $v_{3\lambda\mu}$ ограничена, далее получим оценки для функции $\tilde{u}_{\lambda\mu}(z)$, рассматривая отдельно случаи $\lambda, \mu \in (0, 1)$ и $\lambda, \mu \in (1, +\infty)$:

$$\begin{aligned} |\tilde{u}_{\lambda\mu}(z)| &\leq \\ &\leq \begin{cases} c_{23} \lambda^{-1/2-\alpha} \mu^{-1/2-\beta}, & 0 < \lambda, \mu < 1, \\ c_{24} \mu^{-4,5} \lambda^{-4,5}, & 1 < \lambda, \mu < +\infty. \end{cases} \end{aligned}$$

Используя полученные оценки функции $\tilde{u}_{\lambda\mu}(z)$, перейдём к оценке соответствующих операторов:

$$\begin{aligned} |B_{\alpha-1/2}^x u| &\leq \begin{cases} c_{25}, & 0 < \lambda, \mu < 1, \\ c_{26}, & 1 < \lambda, \mu < +\infty, \end{cases} \\ |B_{\beta-1/2}^y u| &\leq \begin{cases} c_{27}, & 0 < \lambda, \mu < 1, \\ c_{28}, & 1 < \lambda, \mu < +\infty, \end{cases} \\ |B_{\gamma-1/2}^z u| &\leq \begin{cases} c_{29}, & 0 < \lambda, \mu < 1, \\ c_{30}, & 1 < \lambda, \mu < +\infty. \end{cases} \end{aligned}$$

Установленная равномерная сходимость операторов на указанных промежутках позволяет заключить о существовании решения задачи DN_∞ , причём данное решение выражается формулой (24).

Литература

- [1] Agostinelli C. Integrazione dell'equazione differenziale $u_{xx} + u_{yy} + u_{zz} + x^{-1}u_x = 0$ e problema analogo a quello di Dirichlet per un campo emisferico // Atti della Accademia Nazionale dei Lincei. 1937. 6(26). – P. 7–8 (Italian).
- [2] Олевский М.Н. Решения задачи Дирихле, относящейся к уравнению $\Delta u + p x_n^{-1} u_{x_n} = f$ для полусферической области // Докл. АН СССР. 1949. Т. 64. № 6. С. 767–770.
- [3] Капилевич М.Б. Об одном уравнении смешанного эллиптического-гиперболического типа // Математический сборник. 1952. № 1. С. 11–38.
- [4] Urinov A.K., Karimov K.T. The unique solvability of boundary value problems for a 3D elliptic equation with three singular coefficients // Russian Mathematics. 2019. Vol. 63, No. 2. P. 61–72.
- [5] Karimov K.T. Boundary Value Problems in a Semi-infinite Parallelepiped for an Elliptic Equation with Three Singular Coefficients // Lobachevskii Journal of Mathematics. 2021. V. 42, No. 3. P. 560–571.

- [6] Urinov A.K., Karimov K.T. Dirichlet problem for an elliptic equation with three singular coefficients // Journal of Mathematical Sciences. 2021. V. 254, No. 6. P. 731–742.
- [7] Urinov A.K., Karimov K.T. The Dirichlet problem for an elliptic equation with three singular coefficients and negative parameters // Journal of Mathematical Sciences. 2023. V. 274, No. 2. P. 285–300.
- [8] Karimov E.T. On a boundary problem with Neumann's condition for 3D singular elliptic equations // Applied Mathematics Letters 23 (2010), 517–522.
- [9] Nieto J.J., Karimov E.T. The Dirichlet problem for a 3D elliptic equation with two singular coefficients // Computers and Mathematics with Applications 62 (2011), 214–224.
- [10] Эргашев Т.Г. Потенциалы двойного и простого слоев для трёхмерного эллиптического уравнения с сингулярным коэффициентом и их применение // Известия вузов. Математика. 2021. № 1. С. 80–95.
- [11] Снеддон И. Преобразования Фурье. М.: ИЛ, 1955. 668 с.
- [12] Диткин В.А., Прудников А.П. Интегральные преобразования и операционное исчисление. М.: Физматгиз, 1961. 524 с.
- [13] Ватсон Г.Н. Теория бесселевых функций. Т. 1. М.: Изд. ИЛ, 1949. 798 с.
- [14] Пикулин В.П., Похожаев С.И. Практический курс по уравнениям математической физики. 2-е изд. М.: МЦНМО, 2004. 208 с.
- [15] Фихтенгольц Г.М. Курс дифференциального и интегрального исчисления. Т. 2. М.: Наука, 1969.
- [16] Киприянов И.А. Сингулярные эллиптические краевые задачи. М.: Наука, Физматлит, 1997. 204 с.
- [17] Капилевич М.Б. Об одном уравнении смешанного эллиптического-гиперболического типа // Математический сборник. 1952. № 1. С. 11–38.
- [18] Лебедев Н.Н. Специальные функции и их приложения. М.: Физматгиз, 1963. 358 с.
- [19] Толстов Г.П. Ряды Фурье. М.: Наука, 1980. 384 с.

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Статистический контроль радоновых предвестников в прямой и обратной задачах краткосрочного прогнозирования землетрясений

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Аннотация: В статье рассматривается методология краткосрочного прогнозирования землетрясений по радоновым предвестникам на основе статистического контроля. В качестве центрального элемента подхода рассматриваются прямая и обратная задачи. Прямая задача заключается в оценивании параметров ожидаемого землетрясения по характеристикам радоновой аномалии, включая максимальную концентрацию радона, максимальную скорость его роста, продолжительность спада и площадь аномальной зоны. Для прямой задачи используются Пуассоновская и логарифмическая модели, позволяющие связать характеристики предвестников с магнитудой и временем землетрясения. Обратная задача, напротив, состоит в восстановлении допустимых характеристик радонового предвестника по известным параметрам землетрясения. Показано, что логарифмическая обратная задача имеет некорректный характер, вследствие чего для получения устойчивых оценок предлагается использовать контрольную карту Шухарта. Выход радонового процесса за контрольные границы используется для статистического выделения аномальной области и ограничения области допустимых значений параметров предвестника. Предложенная последовательность включает постановку прямой и обратной задач, выявление некорректности обратной задачи, статистическую регуляризацию с помощью контрольных карт, определение начала аномалии и получение устойчивых оценок характеристик предвестника. Результаты формируют основу для дальнейшей верификации методики на исторических данных радонового и сейсмического мониторинга.

Ключевые слова: землетрясение, краткосрочное прогнозирование, радоновый предвестник, статистический контроль, контрольная карта Шухарта, прямая задача, обратная задача, Пуассоновская модель, логарифмическая модель, радоновая аномалия.

Abstract: In this paper, a methodology for short-term earthquake forecasting based on radon precursors and statistical process control is considered. The approach is formulated through two interconnected problems: a forward problem and an inverse problem. The forward problem consists in estimating parameters of an expected earthquake from characteristics of a radon anomaly, including maximum radon concentration, maximum growth rate, decay duration, and the area of the anomalous zone. Poisson and logarithmic models are used to relate precursor characteristics to earthquake magnitude and occurrence time. The inverse problem consists in estimating admissible precursor characteristics from known earthquake parameters. It is shown that the inverse problem in the logarithmic model has an ill-posed character; therefore, a Shewhart control chart is proposed as a statistical regularization mechanism. Exceeding the control limits is used to identify the anomalous region and to restrict the admissible range of precursor parameters. The resulting methodology includes formulation of the forward and inverse problems, identification of the ill-posedness of the inverse problem, statistical regularization using control charts, determination of anomaly onset, and obtaining stable estimates of precursor characteristics. The proposed framework provides a basis for further validation using historical radon and seismic monitoring data.

Keywords: earthquake, short-term forecasting, radon precursor, statistical control, Shewhart control chart, forward problem, inverse problem, Poisson model, logarithmic model, radon anomaly.

Аннотация: Maqolada radon prekursorlari va statistik nazorat asosida zilzilalarni qisqa muddatli prognoz qilish metodologiyasi ko'rib chiqiladi. Yondashuv ikkita o'zaro bog'liq masala — to'g'ri va teskari masalalar asosida qurilgan. To'g'ri masala radon anomaliyasining maksimal konsentratsiyasi, maksimal o'sish tezligi, pasayish davomiyligi va anomaliya maydoni kabi ko'rsatkichlari asosida kutilayotgan zilzilaning parametrlarini baholashdan iborat. Bunda prekursorlar xususiyatlarini zilzila magnitudasi va sodir bo'lish vaqti bilan bog'lash uchun Puasson va logarifmik modellar qo'llaniladi. Teskari masala ma'lum zilzila parametrlariga ko'ra radon prekursorining mumkin bo'lgan xususiyatlarini tiklashdan iborat. Logarifmik modeldagi teskari masala noto'g'ri qo'yilgan masala xususiyatiga ega bo'lgani sababli, barqaror baholarni olish uchun Shewhart nazorat kartasidan statistik regularizatsiya vositasi sifatida foydalanish taklif etiladi. Nazorat chegaralaridan chiqish radon jarayonidagi anomal sohani ajratish va prekursor parametrlarining mumkin bo'lgan qiymatlari oralig'ini cheklash uchun ishlatiladi. Metodologiya to'g'ri va teskari masalalarni qo'yish, teskari masalaning noto'g'ri qo'yilganligini aniqlash, nazorat kartalari yordamida statistik regularizatsiya qilish, anomaliya boshlanish vaqtini aniqlash va prekursor xususiyatlarining barqaror baholarini olish bosqichlarini o'z ichiga

oladi. Taklif etilgan yondashuv radon va seysmik monitoringning tarixiy ma'lumotlarida keyingi tekshiruvlar uchun metodik asos yaratadi.

Kalit so'zlar: zilzila, qisqa muddatli prognoz, radon prekursori, statistik nazorat, Shewhart nazorat kartasi, to'g'ri masala, teskari masala, Puasson modeli, logarifmik model, radon anomaliasi.

I. Введение

Краткосрочное прогнозирование землетрясений по геофизическим предвестникам представляет собой задачу, в которой необходимо связать наблюдаемые изменения контролируемого параметра с будущими характеристиками сейсмического события [1-4]. В рассматриваемой работе таким параметром является концентрация радона, изменение которой анализируется как случайный процесс. Радоновые аномалии рассматриваются не изолированно, а совместно со статистическими характеристиками процесса и результатами контрольного мониторинга. Такой подход позволяет перейти от простого обнаружения аномалии к количественной оценке параметров предполагаемого события.

В задачах прогноза по радоновым предвестникам естественно возникают две взаимосвязанные задачи. Первая — прямая: по наблюдаемым характеристикам радоновой аномалии определить параметры будущего землетрясения. Вторая — обратная: по известным или заданным параметрам землетрясения определить допустимые характеристики радонового предвестника. В исходной работе эти две постановки связываются с Пуассоновской и логарифмической моделями, а контрольные карты используются для решения проблемы устойчивости обратной задачи.

Цель статьи состоит в систематизации предложенной методологии статистического контроля радоновых предвестников и в объединении прямой и обратной постановок в единую схему краткосрочного прогноза землетрясений.

II. Методы исследования

В качестве исходного объекта рассматривается случайный процесс $X(t)$, описывающий изменение концентрации

радона во времени. Из него выделяются характеристики радоновой аномалии: $R_{\max}$ — максимальная концентрация радона; $v_{\max}$ — максимальная скорость роста радона; τ — временной интервал от пика до последующего спада; S — площадь зоны радоновой аномалии; t_p — время достижения максимального пика; $T_e = t_p + \tau$ — время землетрясения. В исходной статье эти характеристики интерпретируются как количественные показатели силы и динамики аномального процесса.

Методологически используются два взаимодополняющих инструмента. Первый — вероятностная модель числа и времени появления аномальных пиков, основанная на Пуассоновском представлении. Второй — логарифмическая регрессионная модель, связывающая магнитуду с нормированными характеристиками радоновой аномалии. Параметры моделей предлагается оценивать по данным предыдущих землетрясений методом наименьших квадратов.

Для статистического выделения аномалии используется контрольная карта Шухарта. В ней выход наблюдаемого процесса за установленные контрольные границы трактуется как статистический сигнал, на основании которого выделяется аномальная область. В рамках данной статьи контрольная карта выполняет не только функцию обнаружения сигнала, но и функцию статистического ограничения при решении обратной задачи.

Таким образом, последовательность метода можно представить как: наблюдение $X(t)$ → статистический контроль → выделение радоновой аномалии → оценивание $R_{\max}$, $v_{\max}$, τ и S → определение магнитуды M и времени T_e в прямой задаче; либо, в обратном направлении, заданные M и T_e → контрольные ограничения →

восстановление допустимых характеристик радонового предвестника.

III. Результаты

3.1. Прямая задача

Прямая задача состоит в определении параметров будущего землетрясения по наблюдаемым характеристикам радонового предвестника. В исходной работе для этой цели рассматриваются Пуассоновская и логарифмическая модели.

В Пуассоновской модели используется число аномальных пиков за рассматриваемый промежуток времени. Пусть N_p — число пиков за время T , K — случайное число пиков от момента сигнала t_p до землетрясения, а a и b — региональные параметры модели. Оценка среднего числа пиков $E(K)$ определяется по экспериментальным данным предыдущих землетрясений по следующей формуле:

$$\hat{E}(K) = \frac{1}{n} \sum_{i=1}^n K_i$$

где:

n — число землетрясений;

K_i — число аномальных пиков между сигналом (t_p) и землетрясением.

Тогда для Пуассоновской модели решение прямой задачи имеет вид:

$$\hat{t} = T \frac{E(K) - N_p}{N_p};$$

$$M = \frac{\log E(K) - a}{b} \quad (1)$$

t_p определяется из диаграммы контрольной карты, причем:

$$T_e \approx t_p + \hat{t} \hat{E}(K)$$

В случае логарифмической модели магнитуда оценивается по формуле:

$$M = a + b \log \left(\frac{R_{max}}{R_0} \right) + c \log \left(\frac{v_{max}}{v_0} \right) + d \log \left(\frac{S}{S_0} \right) \quad (2)$$

где:

R_0 — фоновый уровень радона;

v_0 — типичная скорость;

S_0 — характерная площадь аномалии.

Коэффициенты a, b, c, d оцениваются по экспериментальным данным предыдущих землетрясений по методу наименьших квадратов.

Например, если: $R_{max} = 60 \frac{\text{Бк}}{\text{м}^3}, v_{max} = 12 \frac{\text{Бк}}{\text{м}^3 \cdot \text{сут}}, S = 1500, R_0 = 20, v_0 = 4, S_0 = 300, a = 3.1, b = 0.7, c = 0.5$. То по формуле (2) получаем: $M \approx 5.9$.

Для времени: $t_p = 25$ апреля, 8:00, $\hat{t} = \frac{1}{4}$ суток, $\hat{E}(K) = 4$.

Тогда:

$$T_e \approx 25 + \left(\frac{1}{4} \right) * 4 = 26 \text{ апреля}$$

Эти численные результаты следует рассматривать как иллюстративный пример применения модели, а не как независимую статистическую валидацию

3.2. Обратная задача

Обратная задача состоит в оценке характеристик радонового предвестника по известным параметрам землетрясения. Отметим, что в Пуассоновской модели обратная задача допускает оценивание, тогда как для логарифмической модели она имеет некорректный характер. Поэтому в качестве механизма статистического ограничения используется контрольные карты Шухарта.

Утверждение. Если радоновый процесс анализируется контрольной картой Шухарта и аномальная область определяется выходом за контрольные границы, тогда обратная задача восстановления параметров предвестников становится статистически ограниченной и допускает устойчивую оценку параметров: R_{max} и v_{max} .

Например, если $M = 6, a = 2, b = 0.8, c = 0.5, d = 0.3, v_{max} = 10, S = 500, v_0 = 2, S_0 = 100, R_0 = 20$, то из (2) находим $R_{max} = 120 \frac{\text{Бк}}{\text{м}^3}$.

Тем самым контрольная карта выполняет функцию регуляризации: вместо поиска единственного решения обратной задачи без ограничений рассматривается область параметров, согласующаяся с наблюдаемой статистической аномалией. Это

позволяет связать известные параметры землетрясения с допустимым диапазоном характеристик радонового процесса.

IV. Обсуждение

Полученные результаты показывают, что основная методическая ценность подхода состоит в объединении вероятностного моделирования и статистического контроля. Пуассоновская модель задаёт механизм оценки временной составляющей через число аномальных пиков, а логарифмическая модель связывает интенсивность радоновой аномалии с магнитудой. Контрольная карта Шухарта, в свою очередь, обеспечивает объективное статистическое правило выделения аномалии и вводит ограничения, необходимые для устойчивого решения обратной задачи.

Важным результатом является переход от традиционной схемы «обнаружение предвестника — прогноз события» к двусторонней постановке. В прямой задаче наблюдаемые $R_{\max}$, $v_{\max}$, τ и S используются для оценки M и T_e . В обратной задаче известные M и T_e задают условия, при которых необходимо определить допустимые характеристики радоновой аномалии. Такое объединение делает методологию пригодной как для прогнозной оценки, так и для последующего анализа согласованности наблюдаемого предвестника с параметрами произошедшего землетрясения.

Вместе с тем представленная методология требует дальнейшей эмпирической проверки. В частности, необходимо на независимых исторических рядах оценить устойчивость коэффициентов моделей, определить региональную воспроизводимость контрольных границ, проверить чувствительность результатов к выбору фонового уровня и количественно оценить ошибки прогнозирования магнитуды и времени. Эти вопросы выходят за пределы приведённого в статье теоретико-методического результата и должны рассматриваться на следующем этапе исследования.

Таким образом, контрольная карта в предлагаемой схеме выступает не только как инструмент мониторинга, но и как элемент регуляризации обратной задачи. Именно это обстоятельство позволяет рассматривать статистический контроль как связующее звено между наблюдением радоновой аномалии и количественным прогнозированием параметров землетрясения.

V. Заключение

В работе систематизирована методология краткосрочного прогнозирования землетрясений по радоновым предвестникам на основе статистического контроля. Сформулированы две взаимосвязанные постановки — прямая и обратная. Для прямой задачи использованы Пуассоновская и логарифмическая модели, позволяющие оценивать время и магнитуду землетрясения по характеристикам радоновой аномалии. Для обратной задачи предложено использовать контрольную карту Шухарта как средство статистического ограничения и регуляризации.

Основной методический результат заключается в представлении процесса в виде последовательности: контроль радонового процесса $\rightarrow$ обнаружение статистической аномалии $\rightarrow$ выделение её характеристик $\rightarrow$ оценка параметров землетрясения либо восстановление допустимых характеристик предвестника. Предложенный подход формирует основу для дальнейшего количественного тестирования на данных регионального мониторинга и для развития многопредвестниковой модели.

Литература

- [1]. Ahmedov S.A., Ahmedov O.S. Inverse problems of short-term earthquake forecasting based on radon precursors using control charts. International Conference “Actual Problems of Stochastic Analysis”, May 12–13, 2026, Tashkent, Uzbekistan. P. 20–24.
- [2]. Zakirov T.Z. Features of the Distribution of Radon Concentration in Groundwater of Certain Seismically Active Zones of Uzbekistan (in

connection with the search for earthquake precursors). Dissertation submitted for the degree of Candidate of Geological and Mineralogical Sciences. Tashkent, 1984.

[3]. Kim J.V. et al. Temporal changes in the indoor radon concentration as an earthquake precursor. International Journal of GEOMATE. 2022. Vol. 22, Issue 93. P. 36–43.

[4]. Akhmedov S.A., Akhmedov O.S., Ablazova K.S. Control Charts in the Analysis of Radon Anomalies and Animal Behavior as Indirect Earthquake Precursors. Bukhara State University Scientific Bulletin. 2025. No. 11. P. 78–87.

Фазовые переходы воды: от термодинамики до статистической механики

UDK 536.42

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Annotatsiya. Ushbu maqolada suvning fazaviy o'tishlari: erish, bug'lanish, uchlik nuqta, kritik nuqta va muzning polimorfizmi ko'rib chiqilgan. Suv anomalionalari, o'ta sovutilgan suvda ikkinchi kritik nuqta mavjudligi haqidagi gipoteza hamda fazaviy o'tishning statistik mexanika va Gibbs o'lchovlari tilidagi qat'iy ta'rifi tahlil qilingan. Haqiqiy suv uchun fazaviy o'tishning qat'iy isboti mavjud emasligi ko'rsatilgan.

Kalit so'zlar: suv, fazaviy o'tish, Gibbs o'lchovi, kritik nuqta, muz polimorfizmi, o'ta sovutilgan suv.

Аннотация. В статье рассмотрены основные фазовые переходы воды (плавление, испарение, тройная и критическая точки, полиморфизм льда), аномалии её свойств и гипотеза второй критической точки в переохлаждённой воде. Обсуждается строгое определение фазового перехода в статистической механике как неединственности гиббсовской меры. Показано, что для реальной воды строгого доказательства фазового перехода не существует, а строгие результаты получены лишь для упрощённых моделей.

Ключевые слова: вода, фазовый переход, мера Гиббса, критическая точка, полиморфизм льда, переохлаждённая вода.

Abstract. The paper considers the main phase transitions of water (melting, vaporization, the triple and critical points, ice polymorphism), the anomalies of its properties and the hypothesis of a second critical point in supercooled water. A phase transition is defined in the language of statistical mechanics as non-uniqueness of a Gibbs measure. It is shown that no rigorous proof of a phase transition exists for real water: rigorous results are known only for simplified models such as the lattice gas, the Widom–Rowlinson model and Kac interactions.

Keywords: water, phase transition, Gibbs measure, critical point, ice polymorphism, supercooled water.

Введение

Фазовый переход — это качественное изменение состояния вещества при плавном изменении внешних параметров (температуры, давления, химического потенциала). С точки зрения термодинамики он проявляется как особенность (неаналитичность) термодинамического потенциала, например потенциала Гиббса $G(T, p)$.

Вода интересна тем, что её «обычные» переходы сопровождаются необычными свойствами: лёд легче жидкости, плотность максимальна не у точки замерзания, а около 4 °С, а сжимаемость и теплоёмкость при охлаждении растут, вместо того чтобы падать. В данной статье рассмотрены основные фазовые переходы воды, их термодинамическое описание, микроскопическая природа аномалий и то, что о них известно строго.

1. Классификация фазовых переходов

По Эренфесту переход n -го рода — это переход, при котором первые $n - 1$ производные G непрерывны, а n -я производная испытывает скачок.

- **Первого рода:** скачком меняются энтропия $S = -\partial G / \partial T$ и объём $V = \partial G / \partial p$. Появляется

скрытая теплота $L = T\Delta S$. К ним относятся плавление, кипение, сублимация.

- **Непрерывные (второго рода):** S и V непрерывны, а скачок или расходимость испытывают вторые производные (теплоёмкость, сжимаемость). Пример — критическая точка жидкость–пар.

Современная классификация делит переходы на переходы первого рода и непрерывные, так как «чистые» скачки второй производной по Эренфесту встречаются редко.

2. Основные фазовые переходы воды

2.1. Плавление и кристаллизация

При нормальном давлении лёд I_h плавится при 273,15 К. Теплота плавления составляет около 6,01 кДж/моль (≈ 334 кДж/кг). Объём при плавлении уменьшается примерно на 9 %, поэтому наклон кривой плавления отрицателен. Это следует из уравнения Клапейрона–Клаузиуса:

$$dp/dT = L / (T\Delta v), \quad (1)$$

Поскольку $\Delta v < 0$ (жидкость плотнее льда), $dp/dT < 0$: повышение давления понижает температуру плавления I_h (примерно на 0,0075 К на атмосферу). Это один из редких случаев среди веществ.

2.2. Испарение и конденсация

При 100 °С и 1 атм теплота испарения составляет около 40,7 кДж/моль (≈ 2257 кДж/кг), то есть на порядок больше теплоты плавления. Причина в том, что при испарении разрываются практически все водородные связи. Зависимость давления насыщенного пара от температуры при не слишком больших интервалах описывается уравнением Клаузиуса–Клапейрона:

$$\ln p \approx -L/(R \cdot T) + \text{const.} \quad (2)$$

2.3. Сублимация и тройная точка

Тройная точка воды (сосуществование льда, жидкости и пара) находится при $T = 273,16$ К и $p \approx 611,7$ Па. Раньше она служила определением кельвина. Согласно правилу фаз Гиббса

$$f = c - \varphi + 2 = 1 - 3 + 2 = 0, \quad (3)$$

то есть в тройной точке нет степеней свободы.

2.4. Критическая точка

Кривая кипения заканчивается критической точкой: $T_c \approx 647,1$ К, $p_c \approx 22,06$ МПа, $\rho_c \approx 322$ кг/м³. Выше неё различие между жидкостью и паром исчезает (сверхкритический флюид). Вблизи критической точки наблюдается универсальное поведение. Параметр порядка $\rho - \rho_c$ ведёт себя как

$$\rho_l - \rho_g \sim (T_c - T)^\beta, \quad \beta \approx 0,326, \quad (4)$$

и принадлежит классу универсальности трёхмерной модели Изинга.

2.5. Полиморфизм льда

Под давлением лёд образует множество кристаллических структур: Ih, Ic, II, III, V, VI, VII, VIII, X, XI и другие, а также аморфные льды (LDA, HDA, VHDA). В кристаллах различают протонно-упорядоченные и разупорядоченные формы. При очень высоких давлениях и температурах предсказаны и обнаружены суперионные фазы, где ионы кислорода образуют решётку, а протоны подвижны. Переходы между многими модификациями льда — фазовые переходы первого рода, а переход «упорядочение–разупорядочение» протонов (например, VII–VIII) ближе к непрерывным.

Таблица 1. Характерные величины для переходов воды (приближённые значения)

Переход	Условия	Скрытая теплота	Род
Плавление Ih	273,15 К, 1 атм	6,01 кДж/моль	I
Испарение	373,15 К, 1 атм	40,7 кДж/моль	I
Тройная точка	273,16 К, 611,7 Па	—	—
Критическая точка	647,1 К, 22,06 МПа	0	непрер.

3. Аномалии воды

Именно аномалии делают воду нетривиальной:

1. **Максимум плотности** около 3,98 °С при атмосферном давлении.
2. **Рост изотермической сжимаемости** κ_T и изобарной теплоёмкости C_p при охлаждении ниже ≈ 46 °С (минимум κ_T) и вблизи 35 °С (минимум C_p).
3. **Уменьшение вязкости при сжатии** в определённом диапазоне температур.

Эти эффекты связывают с конкуренцией двух типов локальной структуры: рыхлой тетраэдрической (с большим числом водородных связей, низкой плотностью и низкой энтропией) и более плотной неупорядоченной.

4. Переохлаждённая вода и гипотеза второй критической точки

Чистая вода может оставаться жидкой при температурах значительно ниже 0 °С. Предел её существования при атмосферном давлении определяется гомогенной нуклеацией и составляет около 235 К. Вблизи этой границы сжимаемость и теплоёмкость резко возрастают. Для объяснения этого роста предложены несколько сценариев:

- **Гипотеза жидкость–жидкостной критической точки** (Пул, Шарчиотти, Эссманн, Стэнли, 1992) [1]. Существуют две жидкие фазы — низкоплотная (LDL) и высокоплотная (HDL), — разделённые линией перехода первого рода, которая заканчивается второй критической точкой в области переохлаждения. От неё в стабильную область отходит линия максимумов корреляционной длины (линия Видома), с которой связаны аномалии.
- **Сценарий «свободной от сингулярностей» модели**: аномалии объясняются без настоящего перехода.
- **Сценарий с предельной температурой устойчивости**.

В компьютерных моделях (например, ST2, TIP4P/2005, mW в некоторых вариантах) переход LDL–HDL наблюдается. Прямая экспериментальная проверка очень трудна, поскольку нужные условия лежат в «ничей земле» между гомогенной нуклеацией и стеклованием. Вопрос остаётся предметом активных исследований [2].

5. Теоретическое описание

5.1. Феноменология

Простейшие описания основаны на теории Ван-дер-Ваальса и теории Ландау. Для перехода жидкость–пар уравнение Ван-дер-Ваальса даёт неустойчивую петлю изотерм, которую заменяют правилом Максвелла (равенство химических потенциалов фаз).

5.2. Решёточные и упрощённые модели

Решёточный газ эквивалентен модели Изинга и описывает переход жидкость–пар качественно. Чтобы получить свойства, специфичные для воды, используют модели с направленными связями: модель Белла–Лависа, Mercedes-Benz, а также монотонную модель mW , где трёхчастичный потенциал Стиллинджера–Вебера поддерживает тетраэдрическую геометрию.

5.3. Молекулярное моделирование

Классические модели (SPC/E, TIP4P/2005, TIP5P) [5], поляризуемые модели, многотельные потенциалы (MB-pol) и нейросетевые потенциалы, обученные на данных первопринципных расчётов, воспроизводят множество характеристик воды, включая положение максимума плотности и кривую плавления. Точность разная: например, температура плавления льда T_h заметно зависит от модели.

6. Строгий подход: меры Гиббса

В математической статистической физике фазовый переход определяется как неединственность гиббсовской меры для данной спецификации γ :

$$|G(\gamma)| > 1. \quad (5)$$

Для моделей, близких к воде, строго доказано следующее [3]:

- **Решёточный газ / модель Изинга:** переход при низких температурах доказан аргументом Пайерлса, а для моделей без симметрии — теорией Пирогова–Синяя.

- **Модель Видома–Роулинсона:** строгое доказательство сосуществования фаз в непрерывной системе (Рюэль).
- **Ядра Каца:** переход жидкость–пар доказан в пределе малой силы и дальнего радиуса (Лебовиц–Пенроуз).

Для реалистичной воды (кулоновские и квантовые взаимодействия, водородные связи) строгого доказательства фазового перехода нет. Нет его даже для перехода жидкость–пар в классической непрерывной модели с потенциалом Леннарда–Джонса, а кристаллизация в трёх измерениях строго не доказана ни для одной непрерывной модели с реалистичным потенциалом. Таким образом, все выводы о воде опираются на эксперимент и численное моделирование.

Заключение

Фазовая диаграмма воды включает классические переходы первого рода (плавление, испарение, переходы между льдами), критическую точку жидкость–пар с универсальным поведением и область переохлаждения, где аномалии указывают на возможную вторую критическую точку. Микроскопическая причина богатства диаграммы — конкуренция между тетраэдрической упорядоченностью, поддерживаемой водородными связями, и плотной упаковкой. Строгая математическая теория охватывает лишь упрощённые модели, поэтому «доказательство» для воды остаётся открытой задачей.

Литература

- [1] Poole P.H., Sciortino F., Essmann U., Stanley H.E. Phase behaviour of metastable water. *Nature* 360 (1992).
- [2] Debenedetti P.G. Supercooled and glassy water. *J. Phys.: Condens. Matter* 15 (2003).
- [3] Georgii H.-O. *Gibbs Measures and Phase Transitions*. de Gruyter.
- [4] Brazhkin V. et al. Обзоры по аномалиям и полиморфизму воды и льда.
- [5] Abascal J.L.F., Vega C. A general purpose model for the condensed phases of water: TIP4P/2005. *J. Chem. Phys.* 123 (2005).

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Talabalarning kollaborativ loyiha kompetentligini baholashning mezon-ko'rsatkichlari va darajalari

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Annotatsiya: Maqolada raqamli ta'lim muhitida talabalarning kollaborativ loyiha kompetentligini baholashning mezon-ko'rsatkichlar tizimi ishlab chiqilgan. To'rtta komponent asosida to'rtta mezon va o'n ikkita ko'rsatkich ajratilib, ularning har biri uchun uch darajali deskriptorlar shakllantirilgan. Delfi metodining ikki turi (21 nafar ekspert, $W = 0,78$) orqali ko'rsatkichlar tarkibi aniqlashtirilgan, 96 nafar talaba ishtirokidagi pilot sinov metodikaning ichki izchilligini tasdiqlagan (Kronbax alfa = 0,86; test-retest korrelyatsiyasi $r = 0,79$). Baholashning to'rt manbali tizimi — o'qituvchi ekspertizasi, o'zaro baholash, platforma analitikasi va o'z-o'zini baholash og'irlik koeffitsientlari bilan birlashtirilib, integral ballni hisoblash formulasi taklif etilgan. Ishlab chiqilgan metodika oliy ta'lim muassasalarida loyihaviy topshiriqlarni obyektiv baholash uchun amaliy vosita vazifasini bajaradi.

Kalit so'zlar, kollaborativ loyiha kompetentligi, mezon-ko'rsatkichlar tizimi, deskriptor, Delfi metodi, o'zaro baholash, o'quv analitikasi, integral ball, oliy ta'lim.

I.KIRISH

Jamoaviy loyihalarni baholash oliy ta'lim amaliyotidagi eng nozik masalalardan biri hisoblanadi. O'qituvchi odatda faqat yakuniy mahsulotni ko'radi va unga bitta baho qo'yadi, holbuki bu mahsulot ortida turli darajadagi hissalar turadi. Natijada guruhda ikki xil salbiy hodisa yuzaga keladi ishning asosiy qismini bajargan talaba adolatsizlik hissini tuyadi, minimal ishtirok etgan talaba esa o'zgalar hisobiga baho oladi. Adabiyotda bu hodisa «ijtimoiy bekorchilik» (social loafing) atamasi bilan ifodalanadi va u kollaborativ ta'limning eng barqaror muammosi sifatida tan olinadi. [11]

Raqamli muhit bu muammoni ikki tomonlama o'zgartiradi. Bir tomondan, masofadan ishlash sharoitida o'qituvchining bevosita kuzatuvchi yo'qoladi va nominal ishtirokni aniqlash yanada qiyinlashadi. Ikkinchi tomondan, raqamli platformalar har bir foydalanuvchining harakatlarini — hujjatga kiritilgan tahrirlarni, izohlarni, vazifalar taxtasidagi o'zgarishlarni avtomatik qayd etadi. Ya'ni raqamli muhit adolatli baholash uchun mutlaqo yangi obyektiv asos yaratadi [12,13], ammo bu asosdan foydalanish uchun tegishli mezon-ko'rsatkichlar tizimi zarur.

Amaldagi baholash amaliyotida esa bunday tizim ko'pincha uchramaydi. Ko'pgina fan dasturlarida jamoaviy topshiriq nazarda tutilgan bo'lsa-da, unda individual hissani hisobga olish mexanizmi ham, jarayonni baholash mezonlari ham ko'rsatilmaydi [3,5]. Baho asosan intuitiv tarzda qo'yiladi, bu esa uning ishonchliligini pasaytiradi va talabalarda

baholash tizimiga bo'lgan ishonchni zaiflashtiradi.

II. Adabiyotlar tahlili va metodologiya.

Mezon-ko'rsatkichlar tizimini shakllantirish uchun bosqichda amalga oshirildi. Birinchi bosqichda kompetentlikning to'rt komponentli tuzilmasi asosida [2,4,16] ko'rsatkichlarning dastlabki ro'yxati tuzildi; unga 19 ta ko'rsatkich kiritildi. Ikkinchi bosqichda ro'yxat Delfi metodining ikki turi orqali aniqlashtirildi [15]. Uchinchi bosqichda yakuniy variant pilot sinovdan o'tkazildi.

Ekspert guruhiga 21 nafar mutaxassis jalb qilindi, pedagogika fanlari bo'yicha ilmiy darajaga ega 8 nafar olim, oliy ta'lim muassasalarining kamida besh yillik tajribaga ega 9 nafar professor-o'qituvchisi hamda ta'lim sifatini baholash sohasidagi 4 nafar mutaxassis. Birinchi turda ekspertlar har bir ko'rsatkichning ahamiyatlilik darajasini besh balli shkalada baholadi; o'rtacha bahosi 3,5 balldan past bo'lgan 5 ta ko'rsatkich chiqarib tashlandi. Ikkinchi turda mazmunan takrorlanuvchi 2 ta ko'rsatkich birlashtirildi va yakuniy ro'yxat 12 ta ko'rsatkichdan iborat bo'ldi. Ekspert fikrlarining muvofiqligi Kendall konkordatsiya koeffitsienti orqali tekshirildi, birinchi turda $W = 0,61$, ikkinchi turda $W = 0,78$ ($\chi^2 = 61,3$; $p < 0,01$), ya'ni kelishuv darajasi ikkinchi turda sezilarli darajada oshdi.

Pilot sinovda ikkita oliy ta'lim muassasasining 96 nafar talabasi ishtirok etdi. Metodikaning ichki izchilligi Kronbax alfa koeffitsienti, barqarorligi esa uch hafta oraliq bilan o'tkazilgan takroriy o'lchov (test-retest)

orqali baholandi. Mazmuniy validlik ekspertlar bahosi asosida hisoblangan mazmun validligi indeksi (CVI) yordamida tekshirildi [14].

III. Natijalar va muhokama.

Kompetentlikning to'rt komponenti har biri uchun bitta yetakchi mezon va uchtdan ko'rsatkich belgilandi. Ko'rsatkichlar shunday tanlandiki, ularning har biri kuzatiladigan yoki hujjatlashtiriladigan faktga tayanadi — bu bahoning subyektivligini kamaytiradi (1-jadval).

1-jadval

Kollaborativ loyiha kompetentligini baholashning mezon-ko'rsatkichlar tizimi

Komponent va mezon	Ko'rsatkichlar	Baholash manbayi
Motivatsion-qadriyatli (hamkorlikka yo'nalganlik)	1. Guruh muhokamalarida ixtiyoriy faollik 2. Qo'shimcha vazifani o'z zimmasiga olish holatlari 3. Manbalarga havola berish madaniyati	Platforma analitikasi; o'zaro baholash
Kognitiv-loyihaviy (loyihaviy tafakkur)	4. Muammoning asoslanganligi va aniqligi 5. Maqsad va vazifalar ierarxiyasining izchilligi 6. Manbalarni tanqidiy tanlash va qiyoslash	Loyiha pasporti ekspertizasi
Operatsion-raqamli (raqamli hamkorlik ko'nikmalari)	7. Umumiy hujjatga kiritilgan mazmunli hissa 8. Vazifalarni taqsimlash va muddatlarga rioya 9. Onlayn muloqotning konstruktivligi	Platforma analitikasi; o'qituvchi kuzatuv
Refleksiv-baholovchi (baholovchi mustaqillik)	10. O'z-o'zini bahoning adekvatligi 11. O'zaro bahoning asoslanganligi 12. Tanqid asosida mahsulotni tuzatish	Refleksiv esse; peer-assessment kartasi

Har bir ko'rsatkich 0 dan 3 ballgacha baholanadi, shunga ko'ra maksimal yig'indi 36 ballni tashkil etadi. Ballarning darajalarga taqsimlanishi ekspert muhokamasi asosida

belgilandi, 0–15 ball — reproduktiv daraja, 16–26 ball — konstruktiv daraja, 27–36 ball — ijodiy-mustaqil daraja (1-rasm).

KOLLABORATIV LOYIHA KOMPETENTLIGI SHAKLLANGANLIGINING DARAJALARI

12 ta ko'rsatkich x 0-3 ball = maksimal 36 ball



1-rasm. Kollaborativ loyiha kompetentligi shakllanganligining darajalari va ularning umumlashtirilgan tavsifi

Daraja deskriptorlari. Baholovchilar o'rtasidagi tafovutni kamaytirish uchun har bir komponent bo'yicha uch darajali deskriptorlar ishlab chiqildi. Deskriptorlar kuzatiladigan xulq-

atvor tilida yozildi, ularda «biladi», «tushunadi» kabi tekshirib bo'lmaydigan iboralar o'rniga aniq harakat tavsifi keltirildi (2-jadval).

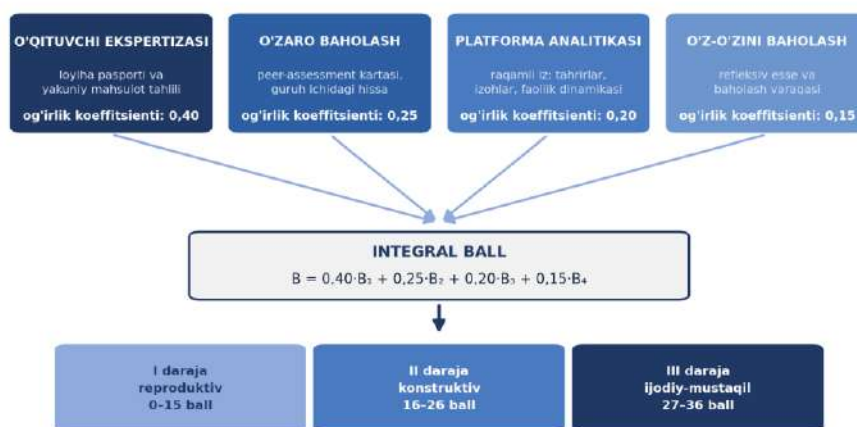
2-jadval

Komponentlar bo'yicha daraja deskriptorlari

Komponent	I daraja (0–1 ball)	II daraja (2 ball)	III daraja (3 ball)
Motivatsion-qadriyatli	Muhokamalarda faqat so'ralganda qatnashadi	Muntazam qatnashadi, o'z vazifasini bajaradi	Tashabbus ko'rsatadi, sheriklarini rag'batlantiradi
Kognitiv-loyihaviy	Tayyor g'oyani qabul qiladi, muammoni asoslamaydi	Muammoni asoslaydi, rejani bosqichlarga bo'ladi	Muqobil yechimlarni qiyoslab, asosli tanlov qiladi
Operatsion-raqamli	Platformadan faqat material olish uchun foydalanadi	Umumiy hujjatda mustaqil ishlaydi, muddatga rioya qiladi	Ish jarayonini tashkil qiladi, sheriklariga texnik yordam beradi
Refleksiv-baholovchi	O'z ishini baholay olmaydi yoki baho keskin oshirilgan	Baholashni mezonlarga tayanib asoslaydi	Tanqidni qabul qilib, mahsulotni maqsadli tuzatadi

Baholashning to'rt manbali tizimi. Ko'rsatkichlarning turli tabiati yagona baholash manbayidan foydalanishga imkon bermaydi. Shu sababli to'rtta manba birlashtirildi va ularning

har biriga og'irlik koeffitsienti berildi (2-rasm). [9,12] Koeffitsientlar qiymati ekspertlar bahosining o'rtacha arifmetigi asosida belgilandi.



2-rasm. Integral ballni shakllantirishning to'rt manbali tizimi

O'qituvchi ekspertizasiga eng yuqori og'irlik (0,40) berilishi uning yagona manba ekanligini anglatmaydi, balki uning faqat mazmuniy sifatni baholashda hal qiluvchi ovozga ega ekanligini bildiradi. O'zaro baholashning ulushi (0,25) guruh ichidagi hissa haqidagi ma'lumot faqat ishtirokchilarning o'zlarida bo'lishi bilan izohlanadi [10]. Platforma analitikasi (0,20) jarayonning

obyektiv qaydini ta'minlaydi. O'z-o'zini baholashga eng past koeffitsient (0,15) berilgan, chunki uning asosiy vazifasi ballni shakllantirish emas, balki refleksiv ko'nikmani rivojlantirishdir.

Metodikaning psixometrik ko'rsatkichlari. Pilot sinov natijalari metodikaning ishonchligini tasdiqladi (3-jadval).

3-jadval

Diagnostik metodikaning ishonchlilik va validlik ko'rsatkichlari (n = 96)

Shkala (komponent)	Ko'rsatkichlar soni	Kronbax alfa	Test-retest (r)	Mazmun validligi indeksi (CVI)
Motivatsion-qadriyatli	3	0,84	0,77	0,86
Kognitiv-loyihaviy	3	0,88	0,81	0,92
Operatsion-raqamli	3	0,87	0,83	0,90
Refleksiv-baholovchi	3	0,82	0,74	0,88
Metodika bo'yicha jami	12	0,86	0,79	0,89

Barcha shkalalar bo'yicha Kronbax alfa qiymati 0,80 dan yuqori bo'lib, bu psixometriyada qabul qilingan me'yorga javob beradi. Eng past ko'rsatkich refleksiv-baholovchi shkalada qayd etildi (0,82), buning sababi — ushbu komponent ko'rsatkichlari ichki holatga eng yaqin turgani va shu bois ularni tashqi kuzatuv orqali qayd etish qiyinroq kechishidir. Test-retest korrelyatsiyasining qoniqarli qiymati ($r = 0,79$) metodika natijalarining vaqt bo'yicha barqarorligini ko'rsatadi.

Muhokama. Ishlab chiqilgan metodikaning asosiy g'oyasi baholash manbalarini ko'paytirish orqali subyektivlikni kamaytirishdan iborat. Bitta manbaga tayangan har qanday baholash tizimi o'sha manbaning cheklovlarini meros qilib oladi, o'qituvchi jarayonni to'liq ko'rmaydi, talabalar o'zaro baholashda do'stona munosabatlar ta'siriga tushadi, platforma esa faollikni qayd etsa-da, uning mazmunli yoki mazmunsizligini farqlamaydi [12]. To'rt manbadan foydalanish bu cheklovlarni o'zaro qoplash imkonini beradi.

O'zaro baholashning ulushi masalasi alohida muhokamaga loyiq. Xalqaro amaliyotda bu ko'rsatkich 0,10 dan 0,40 gacha o'zgaradi [9,10]. Juda past qiymat o'zaro baholashni formal jarayonga aylantiradi va talabalar unga jiddiy yondashmaydi. Juda yuqori qiymat esa guruh ichidagi kelishuvga — barchaning bir-biriga maksimal baho qo'yishiga yo'l ochadi. Pilot sinov davomida 0,25 qiymati optimal deb topildi, bu bahoning ta'sirini sezilarli darajada saqlab qoladi, ammo yakuniy natijani hal

qilmaydi. Bundan tashqari, o'zaro baholash kartasida har bir bahoni qisqa matnli izoh bilan asoslash talabi joriy etildi, bu esa kelishuv riskini sezilarli darajada kamaytirdi.

Platforma analitikasidan foydalanishda ikkita chegaradan chiqmaslik muhim [13]. Birinchisi — miqdorni sifat bilan aralashtirib yubormaslik, ko'p sonli qisqa izohlar qoldirgan talaba bitta chuqur tahliliy fikr bildirgan talabadan faolroq ko'rinadi, holbuki mazmuniy hissassi kamroq bo'lishi mumkin. Shu sababli 7-ko'rsatkich «mazmunli hissa» tarzida shakllantirilgan va uni baholashda o'qituvchi tahrirlar tarixini sifat jihatidan ko'rib chiqadi. Ikkinchi chegara — nazorat vositasiga aylanib ketmaslik, talabalarga analitikaning qaysi parametrlari hisobga olinishi oldindan ochiq e'lon qilinadi, aks holda kuzatilayotganlik hissi hamkorlik muhitiga salbiy ta'sir ko'rsatadi.

Deskriptorlarni kuzatiladigan xulq-atvor tilida yozish tanlovi ham amaliy ahamiyatga ega. Pilot sinov davomida dastlab «guruh maqsadini tushunadi» tipidagi formulirovkalar sinab ko'rildi va ular baholovchilar o'rtasida sezilarli tafovut keltirib chiqardi. Ularni «muhokamada guruh maqsadiga havola qiladi» kabi kuzatiladigan harakat tavsifiga almashtirgandan so'ng baholovchilararo kelishuv ko'rsatkichi oshdi. Bu tajriba deskriptor sifati metodikaning ishonchliligiga to'g'ridan-to'g'ri ta'sir qilishini ko'rsatadi [6,7].

IV.Xulosa.

Kollaborativ loyiha kompetentligini baholash to'rtta mezon va o'n ikkita ko'rsatkich asosida

amalga oshirilishi maqsadga muvofiq; ko'rsatkichlarning har biri kuzatiladigan yoki hujjatlashtiriladigan faktga tayanishi kerak.

Uch daraja — reproduktiv, konstruktiv va ijodiy-mustaqil darajalar 36 balli shkalada 0–15, 16–26 va 27–36 ball oraliqlari bilan chegaralanadi.

Deskriptorlarni kuzatiladigan xulq-atvor tilida shakllantirish baholovchilar o'rtasidagi tafovutni sezilarli darajada kamaytiradi.

Adabiyotlar

- [1]. O'zbekiston Respublikasi Prezidentining 2019-yil 8-oktabrdagi PF-5847-son «O'zbekiston Respublikasi oliy ta'lim tizimini 2030-yilgacha rivojlantirish konsepsiyasini tasdiqlash to'g'risida»gi Farmoni // Qonun hujjatlari ma'lumotlari milliy bazasi. – Toshkent, 2019.
- [2]. Begimqulov U.Sh. Pedagogik ta'lim jarayonlarini axborotlashtirishni tashkil etish va boshqarish nazariyasi va amaliyoti, Ped. fan. dokt. ... diss. – Toshkent, 2007. – 280 b.
- [3]. Muslimov N.A., Usmonboyeva M.H., Sayfurov D.M., To'rayev A.B. Innovatsion ta'lim texnologiyalari, o'quv-metodik qo'llanma. – Toshkent, «Sano-standart» nashriyoti, 2015. – 208 b.
- [4]. Muhamedov G'I. Raqamli ta'lim muhitini loyihalashtirish pedagogik asoslari // Zamonaviy ta'lim. – Toshkent, 2021. – № 4. – B. 12–19.
- [5]. Ishmuhamedov R.J., Yo'ldoshev M. Ta'lim va tarbiyada innovatsion pedagogik texnologiyalar. – Toshkent, «Nihol» nashriyoti, 2013. – 279 b.
- [6]. Xodjayev B.X. Pedagogika nazariyasi va tarixi. – Toshkent, «Sano-standart» nashriyoti, 2017. – 380 b.
- [7]. Sharipov Sh.S. Kasbiy ta'lim jarayonida o'quvchilar ijodkorlik qobiliyatlarini rivojlantirishning uzviyligi. – Toshkent, Fan, 2012. – 168 b.
- [8]. Cronbach L.J. Coefficient alpha and the internal structure of tests // Psychometrika. – 1951. – Vol. 16, No. 3. – P. 297–334.
- [9]. Falchikov N., Goldfinch J. Student peer assessment in higher education, A meta-analysis comparing peer and teacher marks // Review of Educational Research. – 2000. – Vol. 70, No. 3. – P. 287–322.
- [10]. Topping K. Peer assessment between students in colleges and universities // Review of Educational Research. – 1998. – Vol. 68, No. 3. – P. 249–276.
- [11]. Karau S.J., Williams K.D. Social loafing, A meta-analytic review and theoretical integration // Journal of Personality and Social Psychology. – 1993. – Vol. 65, No. 4. – P. 681–706.
- [12]. Strijbos J.W. Assessment of (computer-supported) collaborative learning // IEEE Transactions on Learning Technologies. – 2011. – Vol. 4, No. 1. – P. 59–73.
- [13]. Siemens G., Long P. Penetrating the fog, Analytics in learning and education // EDUCAUSE Review. – 2011. – Vol. 46, No. 5. – P. 30–40.

Integral ball to'rtta manba asosida shakllanadi, o'qituvchi ekspertizasi (0,40), o'zaro baholash (0,25), platforma analitikasi (0,20) va o'z-o'zini baholash (0,15); bunday taqsimot manbalarining cheklovlarini o'zaro qoplash imkonini beradi.

Metodikaning psixometrik ko'rsatkichlari qoniqarli darajada, Kronbax alfa = 0,86, test-retest korrelyatsiyasi $r = 0,79$, mazmun validligi indeksi CVI = 0,89.

[14]. Polit D.F., Beck C.T. The content validity index, Are you sure you know what's being reported? // Research in Nursing & Health. – 2006. – Vol. 29, No. 5. – P. 489–497.

[15]. Hsu C.C., Sandford B.A. The Delphi technique, Making sense of consensus // Practical Assessment, Research & Evaluation. – 2007. – Vol. 12, No. 10. – P. 1–8.

[16]. Johnson D.W., Johnson R.T. An educational psychology success story, Social interdependence theory and cooperative learning // Educational Researcher. – 2009. – Vol. 38, No. 5. – P. 365–379.

Taqqoslamalarni matematikaning boshqa sohalaida qo'llanilishi.

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Annotatsiya. Ushbu maqolada matematikaning bir nechta sohalaridagi ba'zi masalalarni Sonlar nazariyasi fanining asosiy bo'limlaridan bo'lgan taqqoslamalar nazariyasi elementlari yordamida yechish usullari batafsil tahlil qilingan. Z_m chegirmalar sinfi to'plamida binar algebraik amallarni kiritish va undagi bir noma'lumli birinchi darajali tenglamalarni yechish masalalari o'rganilgan. Chiziqli taqqoslamalarni yechishning tanlash, sonlarning EKUBi hamda koeffitsiyentlarni o'zgartirish kabi zamonaviy va qulay usullari keltirilgan. Mazkur usullarni amalda qo'llash orqali butun sonlarga oid, diofant tenglamalari, funksiyalarning butun qiymat qabul qilishi hamda berilgan raqamlar yordamida ko'p xonali sonlarni yasashga doir murakkab matematik masalalarning ajoyib va sodda yechimlari ko'rsatib berilgan.

Kalit so'zlar: Chegirmalar sinfi, chiziqli taqqoslama, EKUB, EKUK, tanlash usuli, koeffitsiyentlarni o'zgartirish.

Abstract. In this work, methods for solving some problems in several areas of mathematics using elements of the theory of congruences, which is one of the main branches of number theory, are analyzed in detail. The introduction of binary algebraic operations in the residue class set Z_m and the problems of solving linear equations with one unknown in this set are studied. Modern and convenient methods of solving linear congruences, such as the trial method, the GCD method, and the coefficient transformation method, are presented. By applying these methods in practice, wonderful and simple solutions to complex mathematical problems related to integers, Diophantine equations, functions taking integer values, and forming multi-digit numbers using given digits are demonstrated.

Keywords: Residue class, linear congruence, GCD, LCM, trial method, coefficient transformation method.

Аннотация. В данной работе подробно анализируются методы решения некоторых задач в нескольких областях математики с использованием элементов теории сравнений, являющейся одним из основных разделов теории чисел. Изучены вопросы введения бинарных алгебраических операций в множестве классов вычетов Z_m и решения линейных уравнений с одним неизвестным в этом множестве. Представлены современные и удобные методы решения линейных сравнений, такие как метод подбора, метод НОД и метод изменения коэффициентов. Путем практического применения этих методов продемонстрированы замечательные и простые решения сложных математических задач, связанных с целыми числами, диофантовыми уравнениями, принятием целых значений функциями и составлением многозначных чисел по заданным цифрам.

Ключевые слова: Класс вычетов, линейное сравнение, НОД, НОК, метод подбора, метод изменения коэффициентов.

I. KIRISH.

$$ax \equiv b \pmod{m}$$

Zamonaviy diskret matematika, axborot xavfsizligi va yuqori unumdorlikka ega hisoblash tizimlarining rivojlanishi arifmetik amallarni chekli algebraik tuzilmalarda optimallashtirish muammosini dolzarb qilib qo'ymoqda. Nemis matematigi K. F. Gauss tomonidan asos solingan taqqoslamalar nazariyasi bugungi kunda sof mavhum algebraik konsepsiya doirasidan chiqib, amaliy matematikaning eng kuchli instrumentlaridan biriga aylandi.

Xususan, kombinator tahlil, hisoblash geometriyasi, xatolarni tuzatuvchi kodlar nazariyasi (Coding Theory) va asimmetrik shifrlash algoritmlari (Kriptografiya) bevosita chegirmalar sinfi halqasidagi xossalarga tayanadi. $Z_m = \{0, 1, 2, \dots, m-1\}$ faktor-to'plami ustida aniqlangan binar algebraik amallar va chiziqli taqqoslamalar:

bu yerda $a, b \in \mathbb{Z}$, x – noma'lum butun son, $m \in \mathbb{N}$ ($m > 1$) esa taqqoslama modulidir. Mazkur tenglamalarning yechimlar fazosi va ularning matematikaning turli fundamental hamda amaliy tarmoqlaridagi integral bog'liqliklarini tadqiq etish, yangi avlod kriptografik barqarorlik modellarini yaratishda muhim ilmiy-amaliy ahamiyat kasb etadi.

II. MATERIAL VA USULLAR.

Z_m chegirmalar halqasi kommutativlik, assotsiativlik va ko'paytirishning qo'shishga nisbatan distributivlik aksiomalarini qanoatlantiradi. Z_6 halqasi misolida binar amallarning strukturaviy xususiyatlarini Keyli matrisalari orqali ifodalash mumkin (1-jadval).

1-jadval. Z_6 chegirmalar halqasining multiplikativ va additiv Keyli matrisalari [7]

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

·	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Matrisalardan ko'rinadiki, $\mathbb{Z}_6$ halqasida noldan farqli elementlarning ko'paytmasi nolga teng bo'lishi holati ($2 \cdot 3 \equiv 0$, $4 \cdot 3 \equiv 0$) kuzatiladi, bu esa halqada *nolning bo'luvchilari* mavjudligini anglatadi. Chiziqli taqqoslamalarni yechish va ularni integral sohalarga tatbiq etishda quyidagi fundamental matematik uslublar qo'llaniladi:

1. Kengaytirilgan Yevklid algoritmi. Agar a va m o'zaro tub bo'lsa, ya'ni $EKUB(a, m) = 1$ bo'lsa, u holda $\mathbb{Z}_m$ halqasida a element uchun yagona a^{-1} (multiplikativ teskari element) mavjud bo'ladi. Buni Bezu chiziqli kombinatsiyasi tengligi orqali ifodalaymiz:

$$au + mv = 1 \Rightarrow au \equiv 1 \pmod{m} \Rightarrow$$

$$a^{-1} \equiv u \pmod{m}$$

U holda $ax \equiv b \pmod{m}$ taqqoslamasining yagona yechimi $x \equiv u \cdot b \pmod{m}$ ko'rinishida aniqlanadi.

2. Eyler va Ferma teoremlari apparati. Agar modul tub son ($m = p$) bo'lsa, Fermaning kichik teoremasidan foydalaniladi: $a^{p-1} \equiv 1 \pmod{p}$. Murakkab modullar va kompyuter algebrasi tizimlari uchun esa Eyler teoremasi tatbiq etiladi:

$$a^{\varphi(m)} \equiv 1 \pmod{m}$$

bu yeda $\varphi(m)$ – m dan katta bo'lmagan va m bilan o'zaro tub bo'lgan natural sonlar sonini aniqlovchi Eyler funksiyasidir.

III. NATIJALAR VA MUHOKAMA

Tadqiqot doirasida chiziqli taqqoslamalar nazariyasining matematikaning turli yo'nalishlaridagi samaradorligi matematik modellashirish va analitik hisoblashlar yordamida isbotlandi.

1. Kriptografik asimmetrik tizimlar (RSA algoritmi).

Axborot xavfsizligida keng qo'llaniladigan RSA shifrlash algoritmining matematik asosi chiziqli taqqoslamalarga tayanadi. Tizimda ochiq kalit e va shifrlash moduli $m = p \cdot q$ (p, q – ulkan tub sonlar) berilgan bo'lsa, maxfiy kalit d quyidagi chiziqli taqqoslamadan aniqlanadi:

$$e \cdot d \equiv 1 \pmod{\varphi(m)}.$$

Bu jarayonda kengaytirilgan Yevklid algoritmi yordamida d yagona yechim (maxfiy kalit) sifatida topiladi. Xabarni shifrlash $C \equiv M^e \pmod{m}$ va deshifrlash $M \equiv C^d \pmod{m}$ funksiyalari orqali modulli darajaga oshirish tamoyilida xavfsiz ishlaydi.

2. Xronologik matematika va taqvim masalalari

Vaqt o'lchovlarining davriyligi (hafta kunlari, oylar, kabisa yillari) $m = 7$, $m = 12$ va $m = 365$ modulli chiziqli taqqoslamalar yordamida mukammal shaklda modellashiriladi. X haftaning n – kuniga mos kelsa, K kundan keyingi muddat:

$$f(K) = (n + K) \pmod{7}$$

funksiyasi orqali chiziqli algoritm ko'rinishida hisoblanadi.

3. Amaliy matematik masalalarning analitik yechimlari.

Quyida oliy algebra, diofant tenglamalari va funksional tahlil kesimidagi chiziqli taqqoslamalar yordamida olingan aniq masalalarning matematik tahlili keltirilgan:

Masala 1 (Integral algebrada invariantlik):

$\frac{11a+2b}{19} \in \mathbb{Z}$ sharti berilganda, $\frac{18a+5b}{19} \in \mathbb{Z}$ ekanligini invariant tahlil usuli bilan isbotlash.

Isboti: Berilgan shartdan $11a + 2b \equiv 0 \pmod{19}$ ekanini kelib chiqadi. Ushbu chiziqli formani $\mathbb{Z}_{19}$ maydonida multiplikativ operator 5 ga ko'paytiramiz:

$$5(11a + 2b) \equiv 55a + 10b \equiv 17a + 10b \equiv 0 \pmod{19}.$$

Ma'lumki,

$$18a + 5b = (17a + 10b) + (a - 5b).$$

Demak, maqsadimiz $(a - 5b) \equiv 0 \pmod{19}$ ekanini ko'rsatish. $\mathbb{Z}_{19}$ halqasida 2 elementining teskarisi $2^{-1} \equiv 10 \pmod{19}$ bo'lgani uchun [1] dastlabki taqqoslamadan $2b \equiv -11a \pmod{19} \Rightarrow b \equiv -110a \equiv 4a \pmod{19}$ kelib chiqadi. Bu qiymatni o'rniga qo'ysak:

$$a - 5b = a - 5(4a) = -19a \equiv 0 \pmod{19}.$$

Isbot yakunlandi.

Masala 2 (Diofant tenglamalari tizimi):

$5x + 4y = 3$ tenglamasini butun sonlar to'plamida yechish.

Yechimi: Ushbu tenglamani $m=4$ moduli bo'yicha chiziqli taqqoslamaga keltiramiz:

$$5x \equiv 3 \pmod{4} \Rightarrow (4 + 1)x \equiv 3 \pmod{4} \Rightarrow x \equiv 3 \pmod{4}.$$

Demak, $x = 3 + 4k$ ($k \in \mathbb{Z}$). Ushbu ifodani

boshlang'ich tenglamaga qo'yib, y ning qiymatini parametrik ko'rinishda aniqlaymiz:

$$\begin{aligned} 5(3 + 4k) + 4y &= 3 \Rightarrow 15 + 20k + 4y = 3 \\ &\Rightarrow 4y = -12 - 20k \Rightarrow \\ y &= -3 - 5k \end{aligned}$$

Yechimlar fazosi:

$$\{(x, y) \mid x = 3 + 4k, y = -3 - 5k, k \in \mathbb{Z}\}.$$

Masala 3. $f(x) = \frac{9x-1}{7}$ funksiyasi $x \in \mathbb{Z}$ ning qanday qiymatlarida $f(x) \in \mathbb{Z}$ bo'lishini aniqlang.

Yechimi. $f(x)$ butun son bo'lishi uchun uning surati maxrajiga qoldiqsiz bo'linishi, ya'ni $9x - 1$ ifodasi 7 soniga karrali bo'lishi zarur va yetarli:

$$9x - 1 \equiv 0 \pmod{7}$$

Ushbu shart $\mathbb{Z}_7$ chegirmalar maydonida chiziqli taqqoslamaga teng kuchlidir:

$$9x \equiv 1 \pmod{7}.$$

Mazkur taqqoslamani yechish uchun koeffitsiyentlarni modulli o'zgartirish usulidan foydalanamiz. Erkin hadga 7 modulining 5 karralisini ($5 \cdot 7 = 35$) qo'shish orqali noma'lum oldidagi 9 koeffitsiyentiga bo'linuvchi shaklni hosil qilamiz:

$$9x \equiv 1 + 35 \pmod{7} \Rightarrow 9x \equiv 36 \pmod{7}$$

$EKUB(9, 7) = 1$ bo'lgani uchun taqqoslamaning ikkala qismini 9 ga qisqartirib, $\mathbb{Z}_7$ halqasidagi yagona fundamental yechimlar sinfini aniqlaymiz:

$$x \equiv 4 \pmod{7} \Rightarrow x = 4 + 7k, \forall k \in \mathbb{Z}.$$

Olingan natijaning invariantligini $k = 0$ va $k = -1$ parametrik qiymatlarda tekshiramiz:

$$1. \quad x_1 = 4 \quad \text{bo'lganda:}$$

$$f(4) = \frac{(9 \cdot 4 - 1)}{7} = \frac{35}{7} = 5 \in \mathbb{Z}$$

$$2. \quad x_2 = -3 \quad \text{bo'lganda:}$$

$$f(-3) = \frac{9 \cdot (-3) - 1}{7} = -\frac{28}{7} = -4 \in \mathbb{Z}$$

Demak, $f(x) = \frac{9x-1}{7}$ funksiya faqat $x \in \{4 + 7k \mid k \in \mathbb{Z}\}$ ekvivalentlik sinfida butun qiymatlar qabul qiladi. Ushbu algoritm an'anaviy usullarga qaraganda ancha sodda va hisoblash jihatidan yuqori unumdorlikka ega.

Masala 4. 523 sonining o'ng tomoniga shunday uchta raqam yozingki, hosil bo'lgan olti xonali son 7, 8, 9 sonlariga qoldiqsiz bo'linsin.

Yechimi. Izlanayotgan olti xonali sonni aniqlash uchun uning o'ng tomoniga yoziladigan uch xonali sonni $n = \overline{xyz}$ ($0 \leq n \leq 999$) deb

belgilaymiz. U holda hosil bo'lgan sonni quyidagi additiv shaklda ifodalash mumkin:

$$523xyz = 523000 + n.$$

Masala shartiga ko'ra, ushbu olti xonali son 7, 8 va 9 sonlariga bir vaqtda bo'linishi kerak. Demak, u ushbu sonlarning eng kichik umumiy karralisiga (EKUK) ham qoldiqsiz bo'linadi:

$$EKUK(7, 8, 9) = 7 \cdot 8 \cdot 9 = 504.$$

Ushbu shart $\mathbb{Z}_{504}$ chegirmalar halqasida quyidagi chiziqli taqqoslamaga ekvivalentdir:

$$523000 + n \equiv 0 \pmod{504}$$

Hisoblashlarni soddalashtirish uchun 523000 sonini 504 moduliga nisbatan eng kichik musbat qoldiq (regulyar chegirma) shaklida approksimatsiyalaymiz:

$$\begin{aligned} 523000 &= 504 \cdot 1037 + 352 \Rightarrow 523000 \\ &\equiv 352 \pmod{504}. \end{aligned}$$

Olingan chegirmani dastlabki taqqoslamaga qo'yib, n parametriga nisbatan chiziqli tenglamani yechamiz:

$$\begin{aligned} 352 + n &\equiv 0 \pmod{504} \Rightarrow n \\ &\equiv -352 \pmod{504} \end{aligned}$$

$\mathbb{Z}_{504}$ halqasidagi invariant musbat chegirmaga o'tish uchun modul qiymatini qo'shamiz:

$$n \equiv -352 + 504 \equiv 152 \pmod{504}.$$

Bundan n ning fundamental yechimlar sinfi kelib chiqadi:

$$n = 152 + 504k, k \in \mathbb{Z}$$

n uch xonali son ($0 \leq n \leq 999$) bo'lganligi sababli, k parametrning faqat $k = 0$ va $k = 1$ qiymatlari chegaraviy shartni qanoatlantiradi:

$$1. \quad k = 0 \quad \text{bo'lganda:} \quad n_1 = 152 \Rightarrow \text{Izlanayotgan birinchi olti xonali son: } 523152$$

$$2. \quad k = 1 \quad \text{bo'lganda:} \quad n_2 = 656 \Rightarrow \text{Izlanayotgan ikkinchi olti xonali son: } 523656$$

ADABIYOTLAR RO'YXATI

- [1] Gauss, K. F. *Disquisitiones Arithmeticae*. Fleischer, Leipzig, 1801.
- [2] Rosen, K. H. *Discrete Mathematics and Its Applications*. 8th ed. New York: McGraw-Hill, 2019.
- [3] Sh. A. Ayupov, B. A. Omirov, A. X. Xudoyberdiyev, F. H. Haydarov. *Algebra va sonlar nazariyasi*. O'quv qo'llanma. Toshkent: "Nodirabegim", 2019.

- [4] Koblitz, N. *A Course in Number Theory and Cryptography*. Graduate Texts in Mathematics, Vol. 114. New York: Springer-Verlag, 1994.
- [5] Rivest, R. L., Shamir, A., Adleman, L. *A Method for Obtaining Digital Signatures and Public-Key Cryptosystems*. Communications of the ACM, 21(2), 120-126, 1978.
- [6] I. Allakov. *Sonlar nazariyasidan misol va masalalar*. O'quv qo'llanma. Termiz: "Surxon-Nashr", 2020.
- [7] Sh. N. Ismoilov. *Sonlar nazariyasi*. O'quv qo'llanma. Toshkent: "Universitet", 2008.
- [8] Cormen, T. H., Leiserson, C. E., Rivest, R. L., Stein, C. *Introduction to Algorithms*. 4th ed. MIT Press, 2022.
- [9]. Stallings, W. *Cryptography and Network Security: Principles and Practice*. 8th ed. London: Pearson, 2020.

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